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Partial Differential Equations and Functional Analysis

The Philippe Clément Festschrift

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Preface

The present volume is dedicated to Philippe Clément on the occasion of his retirement in December 2004. It has its origin in the workshop "Partial Differential Equations and Functional Analysis" (Delft, November 29–December 1, 2004) which was held to celebrate Philippe's profound contributions in various areas of Mathematical Analysis.

The articles presented here offer a panorama of current developments in the theory of partial differential equations as well as applications to such diverse areas as numerical analysis of PDEs, Volterra equations, evolution equations, H^∞-calculus, elliptic systems, mathematical physics, and stochastic analysis. They reflect Philippe's interests very well and indeed several of the authors have collaborated with him in the course of his career.

The editors gratefully acknowledge the financial support of the Royal Netherlands Academy of Arts and Sciences, the Netherlands Organization for Scientific Research, and the Thomas Stieltjes Institute for organizing the workshop. They also thank Thomas Hempfling for the pleasant collaboration during the preparation of this volume.

Last but not least the editors, all members of his former group, thank Philippe for his constant inspiration and for sharing his enthusiasm in mathematics with them.

The Editors

This volume is dedicated to *Philippe Clément*
on the occasion of his retirement.

Philippe Clément: Curriculum Vitae

Philippe Clément, born on 9 January 1943 in Billens, Switzerland, started his study in Physics at the Ecole Polytechnique de l' Université de Lausanne (now EPFL) in 1962 and obtained the degree of Physicist-Engineer in 1967. During that period he discovered that his true interest was much more in Mathematics and he obtained the License des Sciences Mathématiques in 1968 from the University of Lausanne. Hereafter he started to work on his Ph.D. thesis in the area of Numerical Analysis at the EPFL with J. Descloux as supervisor. He defended his thesis, "*Méthode des éléments finis appliquée à des problèmes variationnels de type indéfini*", in February 1974. Some of the results were published in his seminal paper "*Approximation by finite element functions using local regularization*" (Rev. Française Automat. Informat. Recherche Opérationelle, RAIRO Analyse Numérique 9, 1975, R-2, 77–84). In this paper he introduced what is nowadays known in the literature as *the Clément-type interpolation operators*, which play a key role in the analysis of adaptive finite element methods.

In the period 1972–74 Philippe was First Assistant at the Department of Mathematics of the EPFL and under the influence of B. Zwahlen he became interested in Nonlinear Analysis. It was a very stimulating and inspiring time and environment for him, in particular, he met at various workshops Amann, Aubin, Da Prato, Grisvard, Tartar and others. The years 1974–77 Philippe continued his mathematical work, supported by the Swiss National Foundation for Scientific Research, in Madison (USA), first as Honorary Fellow at the Mathematics Department, later as a Research Staff Member at the Mathematics Research Center, of the University of Wisconsin. In that period he came into contact with Crandall and Rabinowitz and worked on nonlinear elliptic problems. Together with Nohel and Londen, he started to be involved in nonlinear Volterra equations.

In 1977 Philippe moved to the University of Technology in Delft, were he was appointed as Associate Professor. In 1980 he became full professor and in 1985 he obtained the Chair in Functional Analysis in Delft. His main areas of interest and research were (and still are) the theory of evolution equations, operator semigroups as well as the Volterra equations and elliptic problems mentioned before. In particular, he was involved in problems concerning maximal regularity and problems related to functional calculus. Philippe is widely recognized for his important contributions in these areas. The very stimulating seminars in Delft on the theory of semigroups have resulted in the book "*One-Parameter Semigroups*" (Clément, Heijmans et al.). In recent years his interests also include stochastic integral equations.

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Harnack Inequality and Applications to Solutions of Biharmonic Equations

Gabriella Caristi and Enzo Mitidieri

This paper is dedicated to our friend Philippe Clément for "not killing birds"

Abstract. We prove Harnack type inequalities for linear biharmonic equations containing a Kato potential. Various applications to local boundedness, Hölder continuity and universal estimates of solutions for biharmonic equations are presented.

1. Introduction

During the last decade considerable attention has been paid to solutions of the time-independent Schrödinger equation

$$-\Delta u = V(x)u,\ x \in \Omega \subseteq \mathbb{R}^N, \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^N$ and the potential V belongs to the Kato class $K^{N,1}_{\text{loc}}(\Omega)$. See, e.g., Aizenman and Simon [1], Zhao [28], [29], [30], Fabes and Strook [8], Chiarenza, Fabes and Garofalo [7], Hinz and Kalf [13], Serrin and Zou [22], Simader [23] and the references therein. Following different approaches these authors have studied the regularity of the solutions and proved a Harnack-type inequality. Such inequality in turn can be used to prove results such as strong maximum principles, removable point singularities, existence of solutions for the Dirichlet problem, Liouville theorems and *universal estimates* on solutions for nonlinear equations.

In the present paper we shall discuss weak solutions of the following fourth-order elliptic equation

$$\Delta^2 u = V\,u,\ x \in \Omega \subseteq \mathbb{R}^N, \tag{1.2}$$

where the potential V is nonnegative in Ω and belongs to $K^{N,2}_{\text{loc}}(\Omega)$ the natural Kato class of potentials associated to the biharmonic operator. See Definition 2.2 in the next section. We point out that Kato classes of potentials associated to polyharmonic operators and some generalizations have been used in a series of works by Bachar et al. see [3], [4] and Maagli et al. [14]. In these papers the authors prove various interesting results including 3G type Theorems and existence of positive solutions for second order and semilinear polyharmonic equations.

Here we are interested in several kinds of results of qualitative nature and mainly some of the consequences that can be deduced from Harnack inequality that we are going to prove during the course. The first one is the regularity problem which includes the local boundedness and the Hölder continuity of solutions. The second kind of results are Harnack-type inequalities. The prototype version of these states: There exist constants $C = C(N)$ and $r > 0$ depending on Ω and norms of V such that all solutions of (1.2) with $u \geq 0$, $-\Delta u \geq 0$ in Ω (i.e., in the sense of distributions in Ω) satisfy

$$\sup_{B_{r/2}} u(x) \leq C \inf_{B_{r/2}} u(x), \tag{1.3}$$

where $B_{r/2}$ denotes any ball contained in Ω.

Our interest in these results was stimulated by studying certain nonlinear biharmonic equations and their isolated singularities (see [5], [25]). For several questions concerning the nature of non-removable singularities and the behavior of a positive solution in a neighborhood of the isolated singularity it is customary to assume that $V \in L^q_{\mathrm{loc}}(\Omega)$ for $q > \frac{N}{4}$.

On the other hand the technique for proving L^p-estimates for all $p < \infty$, relating the $\sup u$ to certain integrals involving the solution of (1.2) as employed by Serrin [20], [21], Stampacchia [26], Trudinger [27] for second order elliptic equations and by Mandras [15] in the context of weakly coupled linear elliptic systems, seems not to be easily extendable for this kind of equations.

In order to derive a local *a priori* majorization for the $\sup u$, we shall follow two different approaches. The first one was introduced by Simader in [23] and it is based on representation formulae of solutions and the $3G$ theorem of Zhao [30], while the latter have been used by Chiarenza et al [7] and its main ingredients are Caccioppoli-type inequalities and maximum principle arguments for the operator $\Delta^2 - V$.

We remark that the Harnack inequality (1.3) admits a rather simple proof in the special case $V \equiv 0$, that is for the biharmonic equation $\Delta^2 u = 0$ in Ω. In fact, due to the special type of mean value formulas for biharmonic functions (see formula $(3.4)_0$ of Simader [24])

$$u(x) = \frac{N+1}{2} \frac{1}{|B_R(x)|} \int_{B_R(x)} u(y) \left[(N+2) - (N+3) \frac{|y-x|}{R} \right] dy$$

it turns out that for $x_0 \in \Omega$ and $B_{2r}(x_0) \subset \Omega$

$$\sup_{B_{r/2}(x_0)} |u(x)| \leq \frac{c(N)}{|B_r(x_0)|} \int_{B_r(x_0)} |u(y)| \, dy,$$

where $c(N) = 2^{N-1}(N+1)(2N+5)$. On the other hand both conditions $u \geq 0$ and $-\Delta u \geq 0$ imply (see [11])

$$\inf_{B_{r/2}(x_0)} u(x) \geq \frac{(2/3)^N}{|B_r(x_0)|} \int_{B_r(x_0)} u(y)\, dy,$$

hence inequality (1.3) follows immediately. We observe that in general there is no hope of obtaining a Harnack inequality for solutions of (1.2) under the only assumption that they are nonnegative. A simple example in this direction is given by $u(x) = x_1^2$ in $\Omega = B_2(0)$. It is interesting to note that the sign conditions $(-\Delta)^m u \geq 0$ in Ω, for $m = 1, \ldots, k-1$, can be already found in the classical book of Nicolescu [18], p. 16, in the context of polyharmonic equations of the type $\Delta^k u = 0$ in Ω, $k \in N$ (see also [2]).

This paper is organized as follows. Section 2 contains few preliminary facts, the proof of the representation formula (see Lemma 2.6) (following Simader [23]) for weak solutions of the problem,

$$(-\Delta)^m u = V\, u,\ x \in \Omega \subseteq \mathbb{R}^N, \tag{1.4}$$

where $V \in K^{N,m}_{\mathrm{loc}}(\Omega), N > 2m, m \geq 2$, and some remarks on Green functions for Schrödinger biharmonic operators. In Section 3 we prove the results on local boundedness and continuity of solutions of (1.2) and a related Harnack inequality (1.3), (see Theorem 3.6). Namely, for each $p \in (0, \infty)$ there exists a constant $C = C(p)$ and $r > 0$ depending on Ω and some local norms of V such that

$$\sup_{B_{r/2}(x_0)} |u(x)| \leq C \left(\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |u(y)|^p\, dy \right)^{1/p}. \tag{1.5}$$

In Section 4 we briefly use the approach by Chiarenza, Fabes and Garofalo [7], adapted to fourth order problems of the type (1.2). The main outcome of this technique is an estimate of the modulus of continuity of the solutions is given in Theorem 4.12 and the Hölder continuity is established in Theorem 4.13 for a class of potentials V which includes those that belongs to the Lebesgue spaces $L^p_{\mathrm{loc}}(\Omega)$ for $p > \frac{N}{4}$. Finally Section 5 is devoted to some consequences of Theorem 4.9. The first application is another proof of Theorem 3.6 while Theorem 5.2 concerns the behavior at infinity of solutions $u \in L^p\left(\mathbb{R}^N\right)$. Also, in Theorem 5.1 and Theorem 5.3 we prove respectively a Harnack inequality of the type (1.3) and a modified version of it in the limiting case $V = O\left(\frac{1}{|x|^4}\right)$ and $\Omega = B_R(0)\backslash\{0\}$.

We conclude the paper with an application to *universal estimates* for a semilinear biharmonic equation in a general domain (see Theorem 5.5). This result can be considered a step towards to the general understanding of the existence of *universal estimates* for solutions of elliptic systems containing non-linearities with growth below the so called *first critical hyperbola* (see [16], [17]).

2. Notation and preliminaries

In the sequel, Ω will denote a nonempty open subset of $\mathbb{R}^N$. If $\Omega_1 \subset \Omega_2 \subseteq \mathbb{R}^N$ are open sets, we write $\Omega_1 \subset\subset \Omega_2$ if and only if $\overline{\Omega}_1$ is compact and $\overline{\Omega}_1 \subset \Omega_2$. For $x \in \mathbb{R}^N$ and $r > 0$ we write $B_r(x)$ to denote the open ball of center x and radius r and $\partial B_r(x)$ to denote its boundary.

In what follows we assume that j is a given nonnegative function in $C_0^\infty(\mathbb{R}^N)$ such that $j(z) = 0$ if $|z| \geq 1$ and:

$$\int_{\mathbb{R}^N} j(x)dx = 1.$$

For any $\epsilon > 0$ we define $j_\epsilon(x) = \epsilon^{-N} j(\epsilon^{-1}x)$.

Definition 2.1. *Given any function $f \in L^q(\Omega)$ with $1 \leq q < \infty$, for any $\epsilon > 0$ we define the* mollified function *by*

$$f_\epsilon(x) = \int_{\mathbb{R}^N} j_\epsilon(x-y)f(y)dy.$$

From the definition it follows that $f_\epsilon \in C^\infty(\Omega) \cup L^q(\Omega)$ and $\|f_\epsilon - f\|_{L^q(\Omega)} \to 0$, as $\epsilon \to 0$.

Definition 2.2. *Given $N > 2m$, the* Kato class $K_{\text{loc}}^{N,m}(\Omega)$ *is the set of functions $V \in L^1_{\text{loc}}(\Omega)$ such that for any compact set $K \subset \Omega$ the quantity*

$$\phi_V(t,K) = \sup_{x \in \mathbb{R}^N} \int_{B_t(x)} \frac{|V(y)|\chi_K(y)}{|x-y|^{N-2m}} dy$$

is finite (here, χ_K denotes the characteristic function of K) and

$$\lim_{t \to 0} \phi_V(t,K) = 0.$$

Example 2.3. Any function satisfying a local Stummel condition. We recall that V satisfies a *local Stummel-condition* if it is measurable in Ω and there exists $\gamma \in (0,4)$ such that for each $\omega \subset\subset \Omega$ there exists a constant $D_\omega > 0$ such that

$$\sup_{x \in \mathbf{R}^N} \int_{\omega \cap B_1(x)} \frac{|V(y)|^2}{|x-y|^{N-4m+\gamma}} dy \leq D_\omega.$$

We claim that if this condition is satisfied, then, $V \in K_{\text{loc}}^{N,m}(\Omega)$. Indeed, for $B_R(x_0) \subset\subset \Omega$ and $0 < t \leq 1$ we have

$$\begin{aligned}
&\int_{B_t(x) \cap B_R(x_0)} \frac{|V(y)|}{|x-y|^{N-2m}} dy \\
&\leq \left(\int_{B_t(x) \cap B_R(x_0)} \frac{|V(y)|^2}{|x-y|^{N-4m+\gamma}} dy \right)^{\frac{1}{2}} \left(\int_{B_t(x)} \frac{dy}{|x-y|^{N-\gamma}} \right)^{\frac{1}{2}} \\
&\leq D_{B_R(x_0)}^{\frac{1}{2}} \left(\frac{\sigma_N}{N\gamma} \right)^{\frac{1}{2}} t^{\frac{\gamma}{2}}.
\end{aligned}$$

Therefore Definition 2.2 is satisfied with $\phi_V(t) = C(D,\gamma)\, t^{\gamma/2}$.

Example 2.4. Any V belonging to $L^{\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha > N/2m$. In fact, in this case if $B_t(x) \subset\subset \Omega$ we have:

$$\begin{aligned}
&\int_{B_t(x)} \frac{V(y)}{|x-y|^{N-2m}} dy \\
&\leq \left(\int_{B_t(x)} |V(y)|^{\alpha} dy \right)^{\frac{1}{\alpha}} \left(\int_{B_t(x)} \frac{1}{|x-y|^{(N-2m)\alpha'}} dy \right)^{\frac{1}{\alpha'}} \\
&\leq \sigma_N^{\frac{1}{\alpha'}} \|V\|_{L^{\alpha}(B_t(x))} t^{2m-\frac{N}{\alpha}}. \qquad (2.1)
\end{aligned}$$

Therefore, Definition 2.2 is satisfied with $\phi_V(t) = C(V)\, t^{2m-\frac{N}{\alpha}}$ for $t > 0$.

Definition 2.5. *We say that $u \in L^1_{\mathrm{loc}}(\Omega)$ is a* distributional solution *of* (1.4), *or that u is a solution in the sense of distributions, if $u \in L^1_{\mathrm{loc}}(\Omega)$ and for any $\psi \in C_0^{\infty}(\Omega)$ we have*

$$\int_{\Omega} u(y)\Delta^m \psi(y) dy = \int_{\Omega} V(y)u(y)\psi(y) dy \qquad (2.2)$$

For any $m \geq 1$, $N > 2m$ and $r > 0$ we set

$$\Phi_{m,N}(r) \;=\; C_{m,N}\, r^{2m-N}, \qquad (2.3)$$

where

$$C_{m,N} = \begin{cases} \dfrac{\Gamma(2-N/2)}{2^{2m-2}(m-1)!\Gamma(m+1-N/2)}, & \text{if } N \text{ is odd,} \\[2ex] (-1)^{m-1} \dfrac{(N/2-m-1)!}{2^{2m-2}(m-1)!(N/2-2)!}, & \text{if } N \text{ is even.} \end{cases}$$

The function $\Phi_{m,N}$ as function of $r = |x|$, for $x \in \mathbb{R}^N \setminus \{0\}$, is *polyharmonic* of degree m and is called the *fundamental solution* of the equation $\Delta^m u = 0$. Here, the constants $C_{m,N}$ are chosen so that $\Delta^p \Phi_{m,N} = \Phi_{m-p,N}$ for $p = 1, \ldots, m-1$, (see [2]). In the sequel, sometimes we will write $\Phi_m(x,y)$ instead of $\Phi_{m,N}(|y-x|)$.

The following *representation formula* holds:

Lemma 2.6. *Let $N > 2m$. If $u \in C_0^{\infty}(\mathbb{R}^N)$, then for any $x \in supp(u)$*

$$u(x) = -\Omega_N \int_{\mathbb{R}^N} \Phi_{m,N}(|y-x|)\Delta^m u(y) dy, \qquad (2.4)$$

where $\Omega_N = ((N-2)\omega_N)^{-1}$ and $\omega_N = 2\pi^{N/2}/\Gamma(N/2)$ is the area of the unit sphere.

Proof. In order to prove (2.4) we apply the second Green identity in $D_\rho = \mathbb{R}^N \setminus \overline{B_\rho(y)}$ successively, add the results and obtain

$$u(x) = \Omega_N \sum_{l=0}^{m-1} \int_{\partial D_\rho} \left(\frac{\partial \Phi_{l+1}(x,y)}{\partial \nu_y} \Delta^l u(y) - \frac{\partial \Delta^l u(y)}{\partial \nu} \Phi_{l+1}(x,y) \right) dy$$
$$-\Omega_N \int_{D_\rho} \Phi_m(x,y) \Delta^m u(y) dy. \tag{2.5}$$

By a standard argument it can be proved that all the nonzero boundary terms tend to 0 as $\rho \to 0$. □

Throughout the paper, for the sake of clarity, we will assume that $m = 2$ and $N > 4$, that is, we shall consider the problem

$$\Delta^2 u = V(x)u, \ x \in \Omega \subseteq \mathbb{R}^N, \ N > 4. \tag{2.6}$$

The proofs of the extensions of the results to the general case can be obtained by obvious modifications.

Lemma 2.7. *Let $g \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ be a distributional solution of $\Delta^2 u = g$. Then, for any $\rho \in C_0^\infty(\Omega)$ the following formula holds:*

$$\begin{aligned} c_N u(x)\rho(x) = & -\int \Phi_2(x,y) g(y) \rho(y) dy \\ & +2 \int u(y) \left(\nabla \Delta \rho(y), \nabla \Phi_2(x,y) \right) dy + 2 \int \Delta^2 \rho(y)\, u(y) \Phi_2(x,y) dy \\ & +2 \int \Delta \left(\nabla \Phi_2(x,y), \nabla \rho(y) \right) u(y) dy + \int \left(\nabla \rho(y), \nabla \Delta \Phi_2(x,y) \right) u(y) dy \\ & +2 \int u(y) \Delta \Phi_2(x,y) \Delta \rho(y) dy - \int u(y) \Delta^2 \rho(y) \Phi_2(x,y) dy, \end{aligned} \tag{2.7}$$

where $c_N = \Omega_N^{-1}$.

Proof. For any $\epsilon > 0$, let u_ϵ be the mollified function of u. Then, given $\rho \in C_0^\infty(\Omega)$ we can apply Lemma 2.6 and get

$$c_N u_\epsilon(x)\rho(x) = -\int_\Omega \Phi_2(x,y) \Delta^2 (u_\epsilon \rho)(y) dy. \tag{2.8}$$

Now, we have

$$\begin{aligned} \Delta^2(u_\epsilon \rho) \quad = & \quad \Delta^2(u_\epsilon)\rho + 2(\nabla \Delta \rho, \nabla u_\epsilon) + 2\Delta\rho\Delta u_\epsilon + 2(\nabla \Delta u_\epsilon, \nabla \rho) \\ + & \quad 2\Delta(\nabla \rho, \nabla u_\epsilon) + u_\epsilon \Delta^2 \rho, \end{aligned} \tag{2.9}$$

and we know that $\Delta^2 u_\epsilon = g_\epsilon$ and that $g_\epsilon \to g$ in $L^1_{\mathrm{loc}}(\Omega)$. To obtain (2.7), first we integrate by parts, taking into account of the fact that all the integrals extended to the boundary of Ω are equal to 0, since $\mathrm{supp}(\rho) \subset \Omega$. Then, we take the limit as $\epsilon \to 0$ and apply Lemma 2.8 below. □

Lemma 2.8. *Let $\Omega \subseteq \mathbb{R}^N$ be bounded, $\rho \in C_0^0(\Omega)$, $g \in L^1_{\mathrm{loc}}(\Omega)$ and $0 < \alpha < N$. Then, if for any $n \in \mathbb{N}$, $\epsilon_n > 0$ and $\epsilon_n \to 0$, as $n \to \infty$, there exists a subsequence, which we still denote by ϵ_n, such that*

$$\int_\Omega g_{\epsilon_n}(y)\rho(y)|y-x|^{-\alpha}dy \to \int_\Omega g(y)\rho(y)|y-x|^{-\alpha}dy, \quad a.e. \text{ in } \mathbb{R}^N.$$

We refer to [23] for its proof.

2.1. Green functions of Schrödinger biharmonic operators

We recall some properties of the Green functions associated to the following boundary value problems for the biharmonic equation on the ball $B_r(0) \subset \mathbb{R}^N$:

$$\Delta^2 u(x) = f(x), \ x \in B_r(0) : \tag{2.10}$$

that is, the Navier boundary value problem (N):

$$u = \Delta u = 0 \text{ on } \partial B_r(0), \tag{2.11}$$

and the Dirichlet boundary value problem (D):

$$u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial B_r(0). \tag{2.12}$$

It is well known that both problems admit a nonnegative Green function on $B_r(0)$, which we denote respectively by $G_r^N(x,y)$ and $G_r^D(x,y)$. Moreover, the following estimates hold:

Lemma 2.9 (3G-Lemma). *Let $G = G_r^N$ or G_r^D. Then, there exists a constant $C_1 > 0$ depending only on the dimension N such that for any ball B of $\mathbb{R}^N$ we have*

$$\frac{G(x,z)G(z,y)}{G(x,y)} \le C_1\left[|x-z|^{4-N} + |z-y|^{4-N}\right],$$

for all x, y, $z \in B$.

The proof of this lemma for the Dirichlet problem is contained in [12], while for the Navier problem it can be straightforwardly obtained by iteration of the 3G-Lemma for the Laplace operator, see [30].

Now, consider instead of (2.10) the Schrödinger biharmonic equation:

$$\Delta^2 u(x) = V(x)u(x) + f(x), \text{ for } x \in B_r(0), \tag{2.13}$$

where V belongs to the natural Kato class $K^{N,2}_{\mathrm{loc}}(B_r(0))$ associated to Δ^2. The following result extends Lemma 2.3 of [6] to the biharmonic case.

Proposition 2.10. *Assume that $V \in K^{N,2}_{\mathrm{loc}}(B_r(0))$. Then, there exists $r_1 > 0$ such that if $0 < r < r_1$, the problems (2.13)–(2.11) and (2.13)–(2.12) admit a nonnegative Green function on $B_r(0)$.*

Proof. Let us consider the Navier boundary value problem: the proof in the other case is similar. Set $A_0(x,y) = G_r^N(x,y)$ and define for $n \geq 1$

$$A_n(x,y) = \int_{B_r} G_r^N(x,z)V(z)A_{n-1}(z,y)dz.$$

We prove by induction that there exists $r_1 > 0$ such that if $0 < r < r_1$ we have

$$|A_n(x,y)| \leq \frac{1}{3^n} A_0(x,y), \quad n \geq 0. \tag{2.14}$$

If $n = 0$, (2.14) holds by definition. Assume that for $n > 0$ (2.14) is true, then we have:

$$|A_{n+1}(x,y)| \leq \frac{1}{3^n} \int_{B_r} G_r^N(x,z)V(z)G_r^N(z,y)dz.$$

By assumption $V \in K_{\mathrm{loc}}^{N,2}(B_r(0))$ and then, we can apply Lemma 2.9 and obtain that

$$|A_{n+1}(x,y)| \leq \frac{1}{3^n} C_1 G_r^N(x,y) \int_{B_r} V(z) \left[|x-z|^{4-N} + |z-y|^{4-N} \right] dz. \tag{2.15}$$

Since $V \in K_{\mathrm{loc}}^{N,2}$, given $\epsilon > 0$ there exists $r_1 > 0$ such that if $0 < r < r_1$, then

$$\int_{B_r(y)} \frac{|V(y)|}{|x-y|^{N-4}} dy < \epsilon.$$

Using this fact in (2.15), we get that

$$|A_{n+1}(x,y)| \leq \frac{1}{3^n} 2C_1 \epsilon G_r^N(x,y),$$

for any $r < r_1$. If we choose $\epsilon = 1/(6C_1)$ we get (2.14). This inequality implies that the series

$$A(x,y) = \sum_{n=0}^{\infty} A_n(x,y)$$

is convergent and that its sum $A(x,y)$ satisfies for all x, $y \in B_r$

$$A(x,y) = G_r^N(x,y) + \int_{B_r} G_r^N(x,z)V(z)A(z,y)dz \tag{2.16}$$

and

$$\frac{1}{2} G_r^N(x,y) \leq A(x,y) \leq 2\, G_r^N(x,z). \tag{2.17}$$

From the last inequality it follows that $A(\cdot,\cdot) \geq 0$. Moreover, applying Δ^2 to both sides of (2.16) for each fixed $y \in B_r$, we have for $x \in B_r$

$$\Delta^2 A(x,y) = \delta_y(x) + V(x)A(x,y)$$

where δ_y is the Dirac function supported at y. Moreover, it is easy to check that the boundary conditions are satisfied. $\square$

Remark 2.11. In particular, from (2.17) it follows that $A(\cdot,\cdot)$ is nonnegative on sufficiently small balls without any assumption on the sign or on the norm of V.

By a similar argument, it can also be proved that if we fix the radius of the ball, for instance, equal to 1, then, there exists a nonnegative Green function for problems (2.10)–(2.11) and (2.10)–(2.12) if $\phi_V(1, B_1)$ is sufficiently small.

3. Local boundedness and continuity of solutions

In this section we will prove the local boundedness and the continuity of distributional solutions of problem (2.6), extending in this way Theorem 2.4 of [23]. The method of proofs is essentially the same.

First of all, we choose $\eta \in C_0^\infty(\mathbb{R})$ such that $0 \le \eta(t) \le 1$, $\eta(t) = 1$ if $t \in [-1/2, 1/2]$ and $\eta(t) = 0$ if $|t| > 1$. Given $\delta > 0$, we set $\eta_\delta(z) = \eta(\delta^{-1}|z|)$, for $z \in \mathbb{R}^N$.

Lemma 3.1. *Assume that $V \in K^{N,2}_{\mathrm{loc}}(\Omega)$. Let $x \in \Omega$ and $0 < \delta < \frac{1}{4}\operatorname{dist}(x, \partial\Omega)$. Then, for any $x \neq z$*

$$\int_\Omega \frac{|V(y)|\eta_\delta(x-y)\eta_\delta(z-y)}{|x-y|^{N-4}|y-z|^{N-4}}dy \le 2^{N-3}\phi_V(\delta, B_{3\delta}(x))\frac{\eta_{4\delta}(x-z)}{|x-z|^{N-4}}. \tag{3.1}$$

Proof. First of all, we observe that if $\sigma = |x - z| > 2\delta$, then it follows that $\eta_\delta(x - y)\eta_\delta(z - y) = 0$. Hence, we can assume that $|x - z| < 2\delta$. Consequently, $\sigma/2 \le \delta$.

Define $\Omega_1 = \{y \in \Omega : |y - x| \le \sigma/2\}$ and $\Omega_2 = \{y \in \Omega : |y - x| \ge \sigma/2\}$, and denote the corresponding integrals by I_1 and I_2. For $y \in \Omega_1$ we have that $|z - y| \ge |z - x| - |x - y| \ge \sigma/2$ and therefore

$$I_1 \le \int_{\Omega_1} \left(\frac{2}{\sigma}\right)^{N-4} \frac{|V(y)|\eta_\delta(x-y)}{|x-y|^{N-4}}dy \le \left(\frac{2}{\sigma}\right)^{N-4} \phi_V(\delta, B_{3\delta}(x)).$$

Similarly,

$$I_2 \le \int_{\Omega_2} \left(\frac{2}{\sigma}\right)^{N-4} \frac{|V(y)|\eta_\delta(y-z)}{|y-z|^{N-4}}dy \le \left(\frac{2}{\sigma}\right)^{N-4} \phi_V(\delta, B_{3\delta}(x)).$$

Since $\eta_{4\delta}(|x - z|) = 1$ for $\sigma \le 2\delta$, these inequalities imply (3.1). □

Lemma 3.2. *Assume that $V \in K^{N,2}_{\mathrm{loc}}(\Omega)$ and that u is a solution of (2.6) in the sense of distributions. Let $x_0 \in \Omega$, $R_0 > 0$ be such that $B_{2R_0}(x_0) \subset\subset \Omega$. Choose $0 < \delta_0 < R_0/4$ such that*

$$\phi_V(\delta) \equiv \phi_V(\delta, B_{R_0}(x_0)) < 2^{2-N}c_N \tag{3.2}$$

for $0 < \delta \le \delta_0$. Then, for $0 < R \le R_0/2$, $0 < \delta \le \delta_0$ and for a.e. $z \in B_R(x_0)$ we have:

$$\int_\Omega \frac{|V(x)||u(x)|\eta_\delta(x-z)}{|x-z|^{N-4}}dx \le 2^{N-4}\delta^{4-N}\|Vu\|_{L^1(B_{R+4\delta}(x_0))} + 2C\delta^{-N}\|u\|_{L^1(B_{R+2\delta}(x_0))}, \tag{3.3}$$

where C depends on N and the choice of η.

Proof. Let $x \in B_{R_0}(x_0)$ and $0 < \delta_0 < R_0/4$. Since $\rho(\cdot) = \eta_\delta(|\cdot - x|) \in C_0^\infty(\Omega)$, we can apply the representation formula of Lemma 2.7 and obtain that:

$$
\begin{aligned}
c_N u(x) = &- \int \Phi_2(x,y)V(y)u(y)\eta_\delta(y-x)dy \\
&+ 2\int u(y)(\nabla\Delta_y\eta_\delta(y-x), \nabla\Phi_2(x,y))dy \\
&+ 2\int \Delta_y^2\eta_\delta(y-x)\, u(y)\Phi_2(x,y)dy \\
&+ 2\int \Delta(\nabla\Phi_2(x,y), \nabla_y\eta_\delta(y-x))u(y)dy \\
&+ \int (\nabla_y\eta_\delta(y-x), \nabla\Delta\Phi_2(x,y))u(y)dy \\
&+ 2\int u(y)\Delta\Phi_2(x,y)\Delta_y\eta_\delta(y-x)dy \\
&- \int u(y)\Delta^2\eta_\delta(y-x)\Phi_2(x,y)dy. \qquad (3.4)
\end{aligned}
$$

Taking into account of the definition of η_δ and of Φ_2 and denoting by χ the characteristic function of the interval $[1/2, 1]$, we get that for any $x \in B_R(x_0)$

$$
\begin{aligned}
|u(x)| \le c_N^{-1} &\int \Phi_2(x,y)|V(y)||u(y)|\eta_\delta(y-x)dy \\
&+ C\delta^{-N}\int |u(y)|\chi(\delta^{-1}|y-x|)dy, \qquad (3.5)
\end{aligned}
$$

where $C = C(N,\eta) > 0$. Using (3.5) in the left-hand side of (3.3), we obtain

$$
\begin{aligned}
&\int \frac{|V(x)||u(x)|\eta_\delta(x-z)}{|x-z|^{N-4}}dx \\
&\le c_N^{-1}\int\int \frac{|V(x)||V(y)||u(y)|\eta_\delta(y-x)\eta_\delta(x-z)}{|x-z|^{N-4}|x-y|^{N-4}}dxdy \\
&\quad + C\delta^{-N}\int \frac{|V(x)|\eta_\delta(x-z)}{|x-z|^{N-4}}\int |u(y)|\chi(\delta^{-1}|y-x|)dxdy. \qquad (3.6)
\end{aligned}
$$

Now, we proceed as in the proof of Lemma 2.3 of [23]. The last integral in (3.6) can be estimated by $\phi_V(\delta)|u|_{L^1(B_{R+2\delta}(x_0))}$. Then, we apply Lemma 2.6 to the first integral and get:

$$
\begin{aligned}
&c_N^{-1}\int |V(y)||u(y)|\int \frac{|V(x)|\eta_\delta(y-x)\eta_\delta(x-z)}{|x-z|^{N-4}|x-y|^{N-4}}dxdy \\
&\le\; c_N^{-1}2^{N-3}\int\int \phi_V(\delta, B_{3\delta}(x))\frac{|V(y)||u(y)|\eta_{4\delta}(y-z)}{|y-z|^{N-4}}dydx. \qquad (3.7)
\end{aligned}
$$

We remark that $\phi_V(\delta, B_{3\delta}(x)) \le \phi_V(\delta)$ and that

$$
\eta_{4\delta}(y-z) = \eta_\delta(y-z) + (\eta_{4\delta}(y-z) - \eta_\delta(y-z)),
$$

hence we get

$$
\begin{aligned}
&c_N^{-1}\int |V(y)||u(y)|\int \frac{|V(x)|\eta_\delta(y-x)\eta_\delta(x-z)}{|x-z|^{N-4}|x-y|^{N-4}}dxdy\\
&\le\ c_N^{-1}2^{N-3}\phi_V(\delta)\int \frac{|V(x)||u(x)|\eta_\delta(x-z)}{|x-z|^{N-4}}dx\\
&+c_N^{-1}2^{N-3}\phi_V(\delta)\int \frac{|V(y)|(\eta_{4\delta}(y-z)-\eta_\delta(y-z))}{|y-z|^{N-4}}|u(y)|dy. \qquad (3.8)
\end{aligned}
$$

Since $\eta_{4\delta}(y-z)-\eta_\delta(y-z))=0$ if $|y-z|\le\delta/2$ and if $|y-z|\ge 4\delta$, the last integral in (3.8) can be estimated by

$$
\left(\frac{\delta}{2}\right)^{4-N}\|Vu\|_{L^1(B_{R+4\delta}(x_0))}. \qquad (3.9)
$$

By assumption, if $0<\delta\le\delta_0$, then

$$
2c_N^{-1}2^{N-3}\phi_V(\delta)\le 1
$$

and hence, from (3.8) and (3.9) the statement follows. □

Theorem 3.3. *Assume that $V\in K^{N,2}_{\mathrm{loc}}(\Omega)$ and that u is a solution of (2.6) in the sense of distributions. Let $x_0\in\Omega$, $R_0>0$ be such that $B_{2R_0}(x_0)\subset\subset\Omega$ and assume that $0<R_1<R_0/2$ is such that*

$$
\phi_V(R_1, B_{2R_0}(x_0))\le(2c_N)^{-1}.
$$

Then, for $0<R\le R_1$, $0<\delta\le\delta_0$ (where δ_0 is defined in Lemma 3.2) and $x\in B_R(x_0)$ the following estimate holds

$$
|u(x)|\le 2^{N-4}\delta^{4-N}\|Vu\|_{L^1(B_{R+4\delta}(x_0))}+C\delta^{-N}\|u\|_{L^1(B_{R+2\delta}(x_0))}, \qquad (3.10)
$$

where C depends only on N and the choice of η.

Proof. The proof follows directly from the proof of Lemma 2.7. □

Corollary 3.4. *Assume that $V\in K^{N,2}_{\mathrm{loc}}(\Omega)$ and that u is a solution of (2.6) in the sense of distributions. Then u is locally bounded.*

The continuity of u follows from its local boundedness. Indeed, in (3.4) all the integrals which include derivatives of η_δ contain no singularity and therefore they define continuous functions. Further, for any $0<\epsilon<\delta$, the function

$$
h_\epsilon(x)=-\int_{\Omega\setminus B_\epsilon(x)}\Phi_2(x,y)V(y)u(y)\eta_\delta(|y-x|)dy
$$

is continuous and converges locally uniformly to the first integral in (3.4) as $\epsilon\to 0$.

In order to prove the Harnack inequality, we need the following result which gives an estimate of the local $\|\cdot\|_\infty$-norm of u in terms of its $\|\cdot\|_1$-norm.

Proposition 3.5. *Assume that* $V \in K^{N,2}_{\mathrm{loc}}(\Omega)$ *and that* u *is a solution of* (2.6) *in the sense of distributions. Let* $x_0 \in \Omega$, $R_0 > 0$ *be such that* $B_{2R_0}(x_0) \subset\subset \Omega$. *Then, there exists* $0 < R_1 \leq R_0/2$ *such that for any* $0 < R < R_1$ *the following estimate holds*

$$\|u\|_{L^\infty(B_R(x_0))} \leq CR^{-N}\|u\|_{L^1(B_{2R}(x_0))}, \tag{3.11}$$

where C *depends only on* N.

Proof. Let $R_1 \leq R_0/2$ be as in Theorem 2.1. Then, for any $0 < R < R_1$:

$$\phi_V(B_{2R}(x_0)) \leq 2^{-1}c_N. \tag{3.12}$$

We shall estimate $\|Vu\|_{L^1(B_{3R/2}(x_0))}$ with the $\|u\|_{L^1(B_{2R}(x_0))}$. To this aim, we take $\rho \in C_0^\infty(\mathbb{R})$ such that $0 \leq \rho \leq 1$, $\rho(\cdot) \equiv 1$ in $[0, 3R/2]$ and $\rho(t) = 0$ if $t \geq 2R$. Further, we assume that $|\rho^{(i)}(t)| \leq t^{-i}$ for $i = 1, 4$ and any t. From Lemma 2.7 we get that for any $x \in B_{2R}(x_0)$

$$\begin{aligned} c_N|u(x)|\rho(x) = &\int |V(y)||u(y)|\Phi_2(x,y)\rho(|x-y|)dy \\ &+ \int |u(y)||R(x,y)|dy, \end{aligned} \tag{3.13}$$

where R is defined in terms of the derivatives of ρ and of Φ_2. Multiplying (3.13) by $|V(x)|$ and integrating over $B \equiv B_{2R}(x_0)$ we get:

$$\begin{aligned} c_N &\int_B |V(x)||u(x)|\rho(x)dx \\ &\leq \int \left[\int \frac{|V(x)|}{|x-y|^{N-4}}dx\right] |V(y)||u(y)|\rho(y)dy \\ &\qquad + \int_B \int_{\frac{3R}{2} \leq |x-y| \leq 2R} |u(y)|\frac{|V(x)|}{|x-y|^{N-4}}dxdy \\ &\leq \phi_V(B_{2R}(x_0)) \int_B |V(x)||u(x)|\rho(x)dx + \phi_V(B_{2R}(x_0)) \int_B |u(x)|dx. \end{aligned} \tag{3.14}$$

From (3.12) and (3.14) we deduce that:

$$\int_B |V(x)||u(x)|\rho(x)dx \leq C\int_B |u(x)|dx, \tag{3.15}$$

which implies that

$$|V(u)|_{L^1(B_{\frac{3R}{2}}(x_0))} \leq c_N^{-1}|u|_{L^1(B_{2R}(x_0))}. \tag{3.16}$$

Now, we apply Theorem 2.1 with $\delta = R/8$. From (3.10) and (3.16), it follows that for any $x \in B_R(x_0)$:

$$\begin{aligned} |u(x)| &\leq 2^{N-4}\delta^{4-N}|V(u)|_{L^1(B_{\frac{3R}{2}}(x_0))} + C\delta^{-N}|u|_{L^1(B_{2R}(x_0))} \\ &\leq CR^N|u|_{L^1(B_{2R}(x_0))}, \end{aligned} \tag{3.17}$$

where $C = C(N)$. □

3.1. The Harnack inequality

Theorem 3.6. *Assume that $V \in K^{N,2}_{\mathrm{loc}}(\Omega)$. Let u be a nonnegative weak solution of (2.6) such that $-\Delta u \geq 0$ in Ω. Then there exist $C = C(N)$ and $R_0 = R_0(V)$ such that for each $0 < R \leq R_0$ and $B_{2R}(x_0) \subset\subset \Omega$ we have*

$$\sup_{B_{R/2}(x_0)} u \leq C \inf_{B_{R/2}(x_0)} u, \tag{3.18}$$

where C independent of V and u.

Proof. Since u is nonnegative and satisfies $-\Delta u \geq 0$ in Ω, then we have (see, e.g., [11])

$$\inf_{B_{R/2}(x_0)} u(x) \geq C(N) \frac{1}{|B_R|} \int_{B_R(x_0)} u(y)\, dy,$$

for all $R > 0$ such that $B_{2R} \subset \Omega$. Then, the conclusion follows from this inequality and (3.11). $\square$

Remark 3.7. The condition $-\Delta u \geq 0$ in Ω is necessary for the validity of the above Theorem as the following example shows: consider the function $u(x) = x_1^2$. It is nonnegative, satisfies $\Delta^2 u = 0$ and $\Delta u = 2$, but (3.18) does not hold for x_0 and any R.

4. Local boundedness and continuity of solutions: an alternative approach

In this section we shall prove The Harnack inequality following the approach of [7], based on L^p estimates. Assume that $m = 2$, $N > 4$ and that $V \in K^{N,2}_{\mathrm{loc}}(\Omega)$.

First of all, we recall the following result from [19]:

Lemma 4.1. *Assume that $V \in K^{N,2}_{\mathrm{loc}}(\Omega)$. Then, for every $\epsilon > 0$ there exists a constant $C = C(\epsilon, V)$ such that*

$$\int_\Omega |V|\, \psi^2 \leq \epsilon \int_\Omega |\Delta\psi|^2 + C \int_\Omega |\psi|^2, \quad \textit{for all}\ \ \psi \in H_0^{2,2}(\Omega). \tag{4.1}$$

For later use we need also the following slightly different inequality. If $t > 0$ $s \in (0,t)$, then:

$$\int_{B_t} |V|\, \psi^2 \leq \epsilon \int_{B_t} |\Delta\psi|^2 + \frac{C(\epsilon, V)}{(t-s)^4} \int_{B_t} |\psi|^2, \quad \text{for all} \quad \psi \in H_0^{2,2}(B_t). \tag{4.2}$$

When $V \in L^{\frac{N}{4-\delta}}(\Omega)$, $\delta \in (0,4)$, a simple proof of the above embedding property (4.2) can be obtained as follows. By the embedding

$$L^{\frac{2N}{N-4}}(\Omega) \cap L^2(\Omega) \hookrightarrow L^{\frac{2N}{N-4+\delta}}(\Omega)$$

and by Sobolev's inequality

$$\|\psi\|_{L^{\frac{2N}{N-4}}(\Omega)} \leq c(N) \|\Delta\psi\|_{L^2(\Omega)}, \text{ for all } \psi \in H_0^{2,2}(\Omega),$$

it follows that for every $\epsilon > 0$

$$\int_\Omega |V|\,\psi^2 \le (t-s)^\delta \|V\|_{L^{\frac{N}{4-\delta}}(\Omega)}\ (t-s)^{-\delta}\|\psi\|^2_{L^{\frac{2N}{N-4+\delta}}(\Omega)}$$
$$\le (t-s)^\delta \|V\|_{\frac{N}{4-\delta}}\left(\epsilon\ \|\psi\|^2_{L^{\frac{2N}{N-4}}(\Omega)} + C(\delta)\,(t-s)^{-4}\,\epsilon^{-\frac{4-\delta}{\delta}}\,\|\psi\|^2_{L^2(\Omega)}\right).$$

Hence, by taking $\epsilon_1 = \epsilon\,(t-s)^\delta \|V\|_{L^{\frac{N}{4-\delta}}(\Omega)}$ we obtain

$$\int_\Omega |V|\,\psi^2 \le \epsilon_1 \int_\Omega |\Delta\psi|^2 + \frac{C}{(t-s)^4}\int_\Omega |\psi|^2, \tag{4.3}$$

where C depends on ϵ_1, δ and $(t-s)^\delta \|V\|_{N/4-\delta}$.

Remark 4.2. As a consequence of (4.1) we deduce that, if $u \in H^{2,2}_{\mathrm{loc}}(\Omega)$, then, $Vu \in L^1_{\mathrm{loc}}(\Omega)$. In fact, let $B_{2r} \subset \Omega$ and let $\varphi \in C_0^\infty(B_{2r})$ be such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on B_r. Since $u\varphi \in H_0^{2,2}(B_{2r})$ and

$$\int_{B_{2r}} |V|\,|u|\,\varphi^2 \le \int_{B_{2r}\cap\{|u|\le 1\}} |V|\,|u| + \int_{B_{2r}\cap\{|u|>1\}} |V|\,|u|\,\varphi^2$$
$$\le \int_{B_{2r}} |V| + \int_{B_{2r}} |V|\ |u\varphi|^2,$$

it follows that $Vu \in L^1(B_r)$.

Definition 4.3. *We say that $u \in H^{2,2}_{\mathrm{loc}}(\Omega)$ is a* weak solution *of* (2.6), *if for any $\psi \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega \Delta u(y)\Delta\psi(y)dy = \int_\Omega V(y)u(y)\psi(y)dy. \tag{4.4}$$

The proof of the local boundedness of any weak solution of (2.6) will be proved through a sequence of lemmas. The first one states a *Caccioppoli-type estimate*([9]). We shall omit its proof

Lemma 4.4. *Let u be a weak solution of* (2.6)*. If $0 < s < t$ and $B_t \subset \Omega$, then there exists a constant $C = C(V)$ independent of u such that*

$$\int_{B_s} |\Delta u|^2 \le \frac{C}{(t-s)^4}\int_{B_t} |u|^2. \tag{4.5}$$

Remark 4.5. We note that also the following estimate holds:

$$\int_{Bs} |\nabla u|^2 \le \frac{C}{(t-s)^2}\int_{B_t} u^2. \tag{4.6}$$

where $C = C(\phi_V)$.

The following lemma is well known see for instance [7].

Lemma 4.6. *Assume that there exist* $\theta \in (0,1)$, $\alpha \geq 0$, $a > 0$ *and a real continuous, non-decreasing and positive function* $I(\cdot)$ *defined on* $(0,1]$ *such that*

$$I(s) \leq a\left(\frac{1}{(t-s)^n} I(t)\right)^{\theta}, \tag{4.7}$$

holds for all $0 < s < t \leq 1$. *Then, for every* $s_1 > 0$ *there exists a constant* $C = C(\theta, n, s_1, a)$ *independent of* I *such that*

$$I(s_1) \leq C. \tag{4.8}$$

The next results provide a kind of reversed Hölder inequality.

Lemma 4.7. *Let* B_{2r} *be a ball contained in* Ω *and let* u *be a weak solution of* (2.6). *Then, there exists a constant* $C(\phi_V)$ *such that*

$$\left(\frac{1}{|B_{r/2}|}\int_{B_{r/2}} u^2\right)^{1/2} \leq C\left(\frac{1}{|B_r|}\int_{B_r} |u|\right). \tag{4.9}$$

Proof. Without restriction we may assume that the center of B_r is the origin. Let $u_r(x) = u(rx)$. Then u_r is a solution of $\Delta^2 u_r = V_r u_r$ in $B_2(0)$, where $V_r(x) = r^4 V(rx)$ satisfies

$$\sup_{x\in B_2}\int_{B_2}\frac{|V_r(y)|}{|x-y|^{N-4}}dy \leq \sup_{0<\alpha<4r}\sup_{w\in\Omega}\int_{B_\alpha(w)}\frac{|V(z)|\,\chi_\Omega(z)}{|w-z|^{N-4}}dz. \tag{4.10}$$

Since $V \in K^{N,2}_{\text{loc}}(\Omega)$, then the right-hand side of (4.10) is bounded and tends to zero as $r \to 0$. We may establish Lemma 4.7 by taking $r = 1$, $\Omega = B_2$, where B_2 is centered at the origin. We may also assume that

$$\int_{B_1} |u| = 1.$$

Define for $0 \leq s \leq 1$:

$$I(s) = \left(\int_{B_s} u^2\right)^{1/2}.$$

Since $L^1(\Omega) \cap L^{\frac{2N}{N-4}}(\Omega) \hookrightarrow L^2(\Omega)$, we get

$$I(s) \leq \left(\int_{B_s} |u|^{\frac{2N}{N-4}}\right)^{\frac{N-4}{2(N+4)}}.$$

By the Sobolev inequality we obtain for $0 \leq s < t \leq 1$

$$I(s) \leq a(\phi_V)\left[\frac{1}{(t-s)^2} I(t)\right]^{\frac{N}{N+4}}. \tag{4.11}$$

The conclusion follows from Lemma 4.6. □

Lemma 4.8. *Let $p \in (0,2)$, $n \geq 0$ and $a > 0$. Let $v \in L^\infty(B_1)$ be nonnegative and*

$$\sup_{B_s} v^2 \leq \frac{a}{(t-s)^n} \int_{B_t} v^2(x)dx,$$

for every $0 < s < t \leq 1$. Then, there exists a constant $C(n,p,a) > 0$ such that

$$\sup_{B_{1/2}} v^p \leq C \int_{B_1} v^p(x)dx.$$

Proof. We may suppose that $v \not\equiv 0$. Putting

$$\gamma = \left(\int_{B_1} v^p(x)dx\right)^{-2/p} \quad \text{and} \quad I(s) = \gamma \int_{B_s} v^2, \text{ for } 0 < s < 1,$$

we note that

$$\sup_{B_{1/2}} v^p \leq [6^n a I(2/3)] \int_{B_1} v^p(x)dx.$$

In order to complete the proof it is sufficient to bound $I(2/3)$ by a constant depending only on n, p and a. To this end let $\alpha = \frac{2}{2-p}$ and $0 < s < t \leq 1$. We obtain

$$\begin{aligned} I(s)^\alpha &= \gamma^\alpha \left(\int_{B_s} v^{2-p}v^p\right)^\alpha \leq \gamma^\alpha \sup_{B_s} v^2 \left(\int_{B_s} v^p\right)^\alpha \\ &\leq \frac{a\gamma^\alpha}{(t-s)^n}\left(\int_{B_s} v^p\right)^\alpha \int_{B_t} v^2 \leq \frac{a}{(t-s)^n} I(t). \end{aligned}$$

Finally, putting $\theta = 1/\alpha$ and $s_1 = 2/3$, the conclusion follows from (4.7) and (4.8). $\square$

The following theorem states the local boundedness of weak solutions of (2.6).

Theorem 4.9. *Let $V \in K^{N,2}_{\text{loc}}(\Omega)$. Then, for each $p \in (0,\infty)$ there exist $C = C(p) > 0$ and $r_1 = r_1(\phi_V) > 0$ such that if u is a weak solution of (2.6), then, for each $0 < R \leq R_1$ and $B_{2R}(x_0) \subset \Omega$, we have*

$$\sup_{B_{R/2}(x_0)} |u(x)| \leq C \left(\frac{1}{|B_r|}\int_{B_R(x_0)} |u(y)|^p \, dy\right)^{1/p}. \tag{4.12}$$

Proof. Without restriction we can assume that $x_0 = 0$. Let $G(x,y)$ be the Green function of the Dirichlet problem for equation (2.13) on B_{2R}, see Proposition 2.10. Choose $R/2 \leq s < t \leq R$ and a function $\varphi \in C_0^\infty(B_{t-(t-s)/4})$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B_{(t+s)/2}$, and $\left|D^k\varphi\right| \leq c/(t-s)^k$, for $k = 1,2$. We claim that for almost all $x \in B_{2R}$ the following identity holds:

$$\begin{aligned} u(x)\varphi(x) &= \int_{B_{2R}} \Delta_y G(x,y)\, u(y)\, \Delta\varphi(y)\, dy - \int_{B_{2R}} G(x,y)\, \Delta u(y)\, \Delta\varphi(y)\, dy \\ &+ 2\int_{B_{2R}} \Delta_y G(x,y)\, (\nabla u(y), \nabla\varphi(y))\, dy - 2\int_{B_{2R}} (\nabla_y G(x,y), \nabla\varphi(y))\Delta u(y)\, dy. \end{aligned}$$

We shall verify this identity assuming that $u \in C^4(B_{2R})$. The general case can be deduced by a standard approximation argument. We have

$$\begin{aligned}\Delta^2(u\varphi) &= u\,\Delta^2\varphi + 2\,(\nabla\Delta\varphi, \nabla u) + 2\Delta\varphi\,\Delta u \\ &\quad + 2\,(\nabla\Delta u, \nabla\varphi) + 2\Delta\,(\nabla\varphi, \nabla u) + \varphi\,\Delta^2 u.\end{aligned}$$

Hence,

$$\begin{aligned}\Delta^2(u\varphi) - Vu\varphi &= u\Delta^2\varphi + 2\,(\nabla\Delta\varphi, \nabla u) + 2\Delta\varphi\,\Delta u \\ &\quad + 2\,(\nabla\Delta u, \nabla\varphi) + 2\Delta\,(\nabla\varphi, \nabla u)\,,\end{aligned}$$

and then,

$$\begin{aligned}u\,\varphi &= \int G\left(u\Delta^2\varphi + 2\,(\nabla\Delta\varphi, \nabla u) + 2\Delta\varphi\,\Delta u + 2\,(\nabla\Delta u, \nabla\varphi) + 2\Delta\,(\nabla\varphi, \nabla u)\right) \\ &= \int \Delta G\,\Delta\varphi\;u - \int G\,\Delta u\,\Delta\varphi - 2\int(\nabla G, \nabla\varphi)\,\Delta u + 2\int \Delta G\;(\nabla u\,, \nabla\varphi)\,.\end{aligned}$$

Let $\mathcal{B}$ denote the set $B_{t-(t-s)/4} \setminus B_{(t+s)/2}$. Then, for almost all $x \in B_{2R}$ we have

$$\begin{aligned}|u(x)\varphi(x)| \;\le\; & \frac{c}{(t-s)^2}\left(\int_{\mathcal{B}} |\Delta_y G(x,y)|^2\,dy\right)^{1/2}\left(\int_{B_t} u^2(y)\,dy\right)^{1/2} \\ & + \frac{c}{(t-s)^2}\left(\int_{\mathcal{B}} |G(x,y)|^2\,dy\right)^{1/2}\left(\int_{\mathcal{B}} |\Delta u(y)|^2\,dy\right)^{1/2} \\ & + \frac{c}{(t-s)}\left(\int_{\mathcal{B}} |\Delta_y G(x,y)|^2\,dy\right)^{1/2}\left(\int_{\mathcal{B}} |\nabla u(y)|^2\,dy\right)^{1/2} \\ & + \frac{c}{(t-s)}\left(\int_{\mathcal{B}} |\nabla_y G(x,y)|^2\,dy\right)^{1/2}\left(\int_{\mathcal{B}} |\Delta u(y)|^2\,dy\right)^{1/2}.\end{aligned}$$

From Lemma 4.4 it follows that:

$$\left(\int_{\mathcal{B}} |\Delta u(y)|^2\,dy\right)^{1/2} \le \frac{c}{(t-s)^2}\left(\int_{B_t} u^2(y)\,dy\right)^{1/2},$$

and

$$\left(\int_{\mathcal{B}} |\nabla u(y)|^2\,dy\right)^{1/2} \le \frac{c}{(t-s)}\left(\int_{B_t} u^2(y)\,dy\right)^{1/2}.$$

Using again Lemma 4.4 and Lemma 4.7, we find for $x \in B_s$

$$\left(\int_{\mathcal{B}} |\Delta_y G(x,y)|^2\,dy\right)^{1/2} \le \frac{c}{(t-s)^{5/2}}\int_{B_t} G(x,y)\,dy,$$

$$\left(\int_{\mathcal{B}} |\nabla_y G(x,y)|^2\,dy\right)^{1/2} \le \frac{c}{(t-s)^{3/2}}\int_{B_t} G(x,y)\,dy,$$

and

$$\left(\int_{\mathcal{B}} |G(x,y)|^2\,dy\right)^{1/2} \le \frac{c}{(t-s)^{1/2}}\int_{B_t} G(x,y)\,dy.$$

Since $\int_{B_t} G(x,y)\,dy$ is bounded, from the above estimates we obtain

$$\sup_{B_s} |u(x)| \leq \frac{c}{(t-s)^5} \left(\int_{B_t} u(y)^2 dy \right)^{1/2}. \tag{4.13}$$

Finally in virtue of Lemma 4.8 and estimate (4.13) we get the conclusion, for $p \in (0,2)$:

$$\sup_{B_{r/2}} |u(x)| \leq C(p) \left(\int_{B_r} |u(y)|^p \, dy \right)^{1/p}.$$

The case $p \in (2,\infty)$ follows applying Hölder inequality. □

The continuity of weak solutions can be proved as in the previous section.

4.1. The Hölder continuity

The proof of Theorem 4.12 below will be based on two lemmas, which are adaptations of the Lemmas 3.1 and 3.2 of [23]. The first one is the following.

Lemma 4.10. *Let $0 < \beta < 1$ and $x_0,\ x \in \mathbb{R}^N$, $r > 0$ be such that*

$$\lambda := \left(\frac{|x - x_0|}{r} \right)^{1-\beta} \leq 1. \tag{4.14}$$

Then, there exists a constant $c > 0$ such that for y with $|y - x_0| \geq 2\, r^{1-\beta} \, |x - x_0|^\beta$ we have

$$\left| \frac{1}{|y-x|^{N-4}} - \frac{1}{|y-x_0|^{N-4}} \right| \leq c \left(\frac{|x-x_0|}{r} \right)^{1-\beta} \frac{1}{|y-x_0|^{N-4}}. \tag{4.15}$$

Proof. Let $y \in \mathbb{R}^N$ be such that

$$|y - x_0| \geq 2\, r^{1-\beta} \, |x - x_0|^\beta. \tag{4.16}$$

Since $|x - x_0| = |x - x_0|^\beta \, |x - x_0|^{1-\beta} \leq \lambda/2 \; |y - x_0|$, it follows that

$$|y - x| \geq |y - x_0| - |x - x_0| \geq \left(1 - \frac{\lambda}{2} \right) |y - x_0| \,.$$

From this inequality we get

$$\begin{aligned} \left| \frac{1}{|y-x|^{N-4}} - \frac{1}{|y-x_0|^{N-4}} \right| &= \big| |y-x_0| - |y-x| \big| \sum_{k=0}^{N-5} |y-x_0|^{k-N+4} \, |y-x|^{-k-1} \\ &\leq \frac{|x-x_0|}{|y-x_0|^{N-3}} \sum_{k=0}^{N-5} \left(\frac{1}{1-\lambda/2} \right)^{k+1}. \end{aligned}$$

Hence, the conclusion follows with $c = \sum_{k=0}^{N-5} 2^k$. □

Now, let $x_0 \in \Omega$, $R > 0$ be such that $B_{4R}(x_0) \subset \Omega$ and let $g \in K^{N,2}_{\text{loc}}(\Omega)$ be nonnegative with $\phi_g(t) = \phi_g(t, B_{2R}(x_0))$. Let $\psi \in C_0^\infty(B_{2R}(x_0))$, $0 \le \psi \le 1$ and

$$J(x) := c_N \int \frac{g(y)\,\psi(y)}{|y-x|^{N-4}} dy, \quad x \in \mathbb{R}^N.$$

We know that $J(\cdot)$ is continuous on $B_{2R}(x_0)$. The next lemma provides an estimate of the local modulus of continuity of J.

Lemma 4.11. *There exists a constant $c > 0$ such that for any $\beta \in (0,1)$ and x such that $|x - x_0| \le r$ we have*

$$|J(x) - J(x_0)| \le c \left(\frac{|x - x_0|}{r} \right)^{1-\beta} \phi_g(2r) + 2\phi_g\left(3\, r^{1-\beta} |x - x_0|^\beta\right). \tag{4.17}$$

Proof. Without loss of generality we may assume that $x_0 = 0$. Choose $0 < \beta < 1$ and let $0 < |x| < r$. It follows that

$$\begin{aligned}|J(x) - J(0)| \;&\le\; c_N \int\!\!\int g(y)\chi_{B_{2r}}(y) \left| |x-y|^{4-N} - |y|^{4-N} \right| dy \\ &\le c_N \int_{B_r \cap \{y : |y| \ge 2r^{1-\beta}|x|^\beta\}} \cdots + c_N \int_{B_r \cap \{y : |y| \le 2r^{1-\beta}|x|^\beta\}} \cdots := J_1(x) + J_2(x).\end{aligned}$$

Now, applying Lemma 4.10 to J_1 we obtain

$$J_1(x) \le c \left(\frac{|x|}{r} \right)^{1-\beta} \int \frac{g(y)\chi_{B_{2r}}(y)}{|y|^{N-4}} dy \le c \left(\frac{|x|}{r} \right)^{1-\beta} \phi_g(2r). \tag{4.18}$$

Consider $J_2(\cdot)$. Since $|y| \le 2r^{1-\beta}|x|^\beta$ and $|x|^{1-\beta} \le r^{1-\beta}$ we deduce that

$$|y - x| \le |x| + |y| \le |x|^\beta \left(|x|^{1-\beta} + 2r^{1-\beta} \right) \le 3r^{1-\beta} |x|^\beta,$$

and then,

$$\begin{aligned} J_2(x) \;&\le c_N \int_{B_{2r} \cap \{y : |y-x| \le 3r^{1-\beta}|x|^\beta\}} \frac{g(y)}{|y-x|^{N-4}} dy \\ &\qquad + \; c_N \int_{B_{2r} \cap \{y : |y| \le 2r^{1-\beta}|x|^\beta\}} \frac{g(y)}{|y|^{N-4}} dy \\ &\le 2\eta_g \left(3r^{1-\beta} |x|^\beta \right).\end{aligned}$$

The conclusion now follows from this inequality and (4.18). $\square$

Theorem 4.12. *Let $x_0 \in \Omega$ and $B_{2R}(x_0) \subset \Omega$. If u is a weak solution of (2.6), then, chosen $0 < \beta < 1$ there exists a constant c such that for $|x - x_0| < r$ we have*

$$\begin{aligned}&|u(x) - u(x_0)| \le \\ &\quad \le\; c \left[\left(\frac{|x - x_0|}{r} \right)^{1-\beta} (\phi_V(2r) + 1) + \phi_V \left(3r \left(\frac{|x - x_0|}{r} \right)^\beta \right) \right] \sup_{B_{3r}(x_0)} |u|.\end{aligned}$$

Proof. Without loss of generality we may assume that $x_0 = 0$ and set $B_{4r} = B_{4r}(0)$. Let $\varphi \in C_0^\infty(B_{2r})$, $0 \le \varphi \le 1$ with $\varphi \equiv 1$ on $B_{3r/2}$ and $\left|D^k\varphi\right| \le c/r^{|k|}$, for $k = 1, 2$. We have

$$\begin{aligned} u(x)\varphi(x) = & \int \Phi_2(x,y)\, V(y)\, u(y)\, \varphi(y)\, dy - \int \Phi_2(x,y)\Delta u(y)\Delta\varphi(y)dy \\ & + \int \Delta_y\Phi_2(x,y)\, u(y)\, \Delta\varphi(y)\, dy - 2\int \nabla_y\Phi_2(x,y)\cdot\nabla\varphi(y)\Delta u(y)\, dy \\ & + 2\int \Delta_y\Phi_2(x,y)\nabla u(y)\cdot\nabla\varphi(y)\, dy = \sum_{i=1}^{5} I_i(x). \end{aligned}$$

Fix $\beta \in (0,1)$ and let $|x| < r$. From this identity we obtain

$$\begin{aligned} u(x) - u(0) = & \int (\Phi_2(x,y) - \Phi_2(0,y)\,)\, V(y)\, u(y)\, \varphi(y) \\ & - \int (\Phi_2(x,y) - \Phi_2(0,y))\, \Delta u(y)\Delta\varphi(y)dy \\ & + \int (\Delta_y\Phi_2(x,y) - \Delta_y\Phi_2(0,y))\; u(y)\Delta\varphi(y) \\ & - 2\int (\nabla_y\Phi_2(x,y) - \nabla_y\Phi_2(0,y))\, \nabla\varphi(y)\, \Delta u(y) \\ & + 2\int (\Delta_y\Phi_2(x,y) - \Delta_y\Phi_2(0,y))\, \nabla u(y)\nabla\varphi(y). \end{aligned} \tag{4.19}$$

Since $Vu \in K^{N,2}_{\mathrm{loc}}(B_{2r})$ and $\phi_{Vu}(t) \le \sup_{B_{2r}} |u| \cdot \phi_V(t)$, we can apply Lemma 4.11 to the first term on the right-hand side of (4.19) and obtain

$$|I_1(x) - I_1(0)| \le \left[c\left(\frac{|x|}{r}\right)^{1-\beta} \eta_V(2r) + 2\eta_V\left(3r^{1-\beta}\,|x|^\beta\right)\right] \sup_{B_{2r}} |u|\,. \tag{4.20}$$

To handle the other terms of (4.19), let us observe that all the derivatives of φ vanish for $|y| \le r$. Define for $t \in [0,1]$ $\phi(t) := |y - tx|^{4-N}$, where $y \in B_{2r} \setminus B_{3r/2}$ and $|x| < r$. From the mean value theorem and $|y - t_0x| \ge |y| - |x| \ge r/2$ it follows that

$$\begin{aligned} \phi(1) - \phi(0) = & \left|\,|y-x|^{4-N} - |y|^{4-N}\right| = \phi'(t_0) \\ & \le (N-4)\,\frac{|x|}{|y - t_0x|^{N-3}} \le (N-4)\,2^{N-3}\frac{|x|}{r^{N-3}}, \end{aligned}$$

where $t_0 \in (0,1)$. Therefore,

$$|\Phi_2(x,y) - \Phi_2(0,y)| \le c\frac{|x|}{r^{N-3}} \le c\left(\frac{|x|}{r}\right)^{1-\beta}\frac{1}{r^{N-4}}. \tag{4.21}$$

The same argument shows also that

$$|\nabla_y \Phi_2(x,y) - \nabla_y \Phi_2(0,y)| \le c \left(\frac{|x|}{r}\right)^{1-\beta} \frac{1}{r^{N-3}}, \tag{4.22}$$

$$|\Delta_y \Phi_2(x,y) - \Delta_y \Phi_2(0,y)| \le c \left(\frac{|x|}{r}\right)^{1-\beta} \frac{1}{r^{N-2}}. \tag{4.23}$$

Now we can estimate the remaining terms of (4.19), that is $|I_k(x) - I_k(0)|$ for $k = 2, \dots, 5$, by using the bounds (4.21), (4.22) and (4.23). We find that

$$|I_2(x) - I_2(0)| \le \frac{c}{r^2} \int |\Phi_2(x,y) - \Phi_2(0,y)|\, |\Delta u(y)|\, dy$$

$$\le c \left(\frac{|x|}{r}\right)^{1-\beta} \frac{1}{r^{N-2}} \int_{B_{2r}} |\Delta u| \le c \left(\frac{|x|}{r}\right)^{1-\beta} r^2 \left(\frac{1}{|B_{2r}|} \int_{B_{2r}} |\Delta u|^2\right)^{1/2},$$

and hence, by Lemma 4.4

$$|I_2(x) - I_2(0)| \le c \left(\frac{|x|}{r}\right)^{1-\beta} \left(\frac{1}{|B_{3r}|} \int_{B_{3r}} |u|^2\right)^{1/2}. \tag{4.24}$$

It is also easy to check that

$$|I_3(x) - I_3(0)| \le c \left(\frac{|x|}{r}\right)^{1-\beta} \left(\frac{1}{|B_{3r}|} \int_{B_{3r}} |u|^2\right)^{1/2}, \tag{4.25}$$

and

$$|I_4(x) - I_4(0)| + |I_5(x) - I_5(0)| \le c \left(\frac{|x|}{r}\right)^{1-\beta} \left(\frac{1}{|B_{3r}|} \int_{B_{3r}} |u|^2\right)^{1/2}.$$

Finally, this inequality together with the estimates (4.20), (4.24) and (4.25) complete the proof. □

In order to conclude that the weak solutions of (2.6) are Hölder continuous we restrict ourselves to a class of potentials V such that

$$\phi_V(t) \le M t^{\alpha} \tag{4.26}$$

for some $\alpha > 0$. Clearly, this class is not empty since, according to Example 2.3, we know that if V satisfies a Stummel condition, then (4.26) holds with $\alpha = \gamma/2$, or if V belong to L^p_{loc} with $p > \frac{N}{4}$, then (4.26) holds with $\alpha = 4 - \frac{N}{p}$.

We point out that if $N \le 3$ all weak solutions $u \in H^{2,2}_{\text{loc}}(\Omega)$ of (2.6) are Hölder continuous by Sobolev's embedding.

Theorem 4.13. *Let $x_0 \in \Omega$ and $B_{4R}(x_0) \subset \Omega$. Let u be a weak solution of* (1.1). *Suppose that there exist $\alpha > 0$ and a constant $M_V = M(V, B_{4R})$ such that* (4.26) *holds. Then, if $|x - x_0| \le R$, we have*

$$|u(x) - u(x_0)| \le c \left\{ [M_V (3R)^{\alpha} + 1]\, R^{-\frac{\alpha}{1+\alpha}} \sup_{B_{3R}(x_0)} |u| \right\} |x - x_0|^{\frac{\alpha}{1+\alpha}}. \tag{4.27}$$

Proof. Since $\phi_V(2r) \le M_V(2R)^\alpha$ and

$$\phi_V\left(3r\left(\frac{|x-x_0|}{R}\right)^\beta\right) \le M_V(3r)^\alpha\left(\frac{|x-x_0|}{R}\right)^{\alpha\beta},$$

the conclusion follows directly from Theorem 4.12 by taking $\beta = \frac{1}{1+\alpha}$. □

5. Applications

In this section we shall present some consequences of Theorem 4.9. The first is another proof of Harnack inequality:

Theorem 5.1. *Assume that $V \in K^{N,2}(\Omega)$. Let u be a nonnegative weak solution of (1.1) such that $-\Delta u \ge 0$ in Ω. Then there exist $C = C(N)$ and $r_0 = r_0(\eta_V)$ such that for each $0 < R \le R_0$ and $B_{2R} \subset \Omega$ we have*

$$\sup_{B_{R/2}} u \le C \inf_{B_{R/2}} u.$$

Proof. Since u is nonnegative and satisfies $-\Delta u \ge 0$ in Ω, then we have (see, e.g., [11])

$$\inf_{B_{r/2}} u(x) \ge C(N)\frac{1}{|B_r|}\int_{B_r} u(y)\,dy,$$

for all $r > 0$ such that $B_{2r} \subset \Omega$. The conclusion follows from this inequality and Theorem 4.9 with $p = 1$. □

Theorem 5.2. *Let $p : 1 \le p < \infty$ and $\Omega = \mathbf{R}^N \setminus \overline{B_R(0)}$. If $u \in L^p(\Omega) \cap H^{2,2}_{\mathrm{loc}}(\Omega)$ is a solution of (1.1) with $V \in K^{N,2}(\Omega)$ nonnegative then:*

$$\lim_{|x|\to\infty} |u(x)| = 0$$

Proof. Without loss of generality we may assume that u is continuous in Ω. By Theorem 4.9, it follows that there exists $r_0 = r_0(\eta_V)$ such that for all $x : |x| \ge R+2$, for each $r \in (0, r_0)$ and $0 < s < t < 1$ we have

$$\sup_{B_{sr}(x)} |u(y)|^p \le \frac{C}{r^N(t-s)^N}\int_{B_{tr}(x)} |u(z)|^p\,dz.$$

Since $u \in L^p(\Omega)$ the result follows. □

The next result concerns the limiting case for a Harnack's theorem near the possible isolated singularity $x = 0$ (compare with Theorem 3.1 of [10]).

Theorem 5.3. *Let $\Omega = B_R(0) \setminus \{0\}$ and let $V \in C_{\mathrm{loc}}(\Omega)$ be nonnegative and such that for all $x \in \Omega$*

$$0 \le V(x) \le \frac{c_1}{|x|^4},\ c_1 > 0.$$

Let $u \in H^{2,2}_{\mathrm{loc}}(\Omega)$ be a solution of

$$\begin{cases} \Delta^2 u = Vu & \text{on } \Omega, \\ u \ge 0,\ -\Delta u \ge 0 & \text{in } \Omega. \end{cases} \tag{5.1}$$

Then there exist two constants C and ϑ_1 depending on c_1 such that for each $0 < \vartheta \le \vartheta_1$ and $0 < \epsilon < \frac{1}{2R}$ we have

$$\sup_{\epsilon \le |x| \le (1+\vartheta)\epsilon} u(x) \le C \inf_{\epsilon \le |x| \le (1+\vartheta)\epsilon} u(x). \tag{5.2}$$

Proof. The proof is a slight modification of the proof of Theorem 4.9. Here, we shall emphasize the dependence of the constants on c_1. If $x_0 \in \Omega$ let $r_0 = \frac{1}{4}|x_0|$ so that $B_{2r_0}(x_0) \subset \Omega$. We claim that for every $p \in (0, \infty)$ there exists positive constants $C = C(p)$ and ϑ_0 depending on c_1 such that for all $x_0 \in \Omega$ and $0 < r \le \vartheta_0 r_0$

$$\sup_{B_{r/2}(x_0)} |u(x)| \le C \left(\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |u(y)|^p \, dy \right)^{1/p}. \tag{5.3}$$

To prove (5.3) we fix $\sigma \in (0, 4)$ and note that

$$\sup_{x \in \Omega} r^\sigma \, \|V\|_{L^{\frac{N}{4-\sigma}}(B_r(x))} = C_1 < \infty$$

where C_1 depends on c_1. This fact together with the embedding property (4.2) yields the Lemma 3.2 for each $B_{sr_0}(x_0) \subset B_{tr_0}(x_0) \subset \Omega$. Clearly, Lemma 4.7 and 4.8 hold true. We have to bound

$$\sup_{x \in B_{2r}(x_0)} \int_{B_{2r}(x_0)} \frac{V(y)}{|x-y|^{N-4}} dy,$$

uniformly with respect to $x_0 \in \Omega$. Indeed, since

$$\sup_{x \in B_{2r}(x_0)} \int_{B_{2r}(x_0)} \frac{V(y)}{|x-y|^{N-4}} dy \le c \left(r_0^\sigma \, \|V\|_{L^{\frac{N}{4-\sigma}}(B_{2r_0}(x_0))} \right) \frac{r^\sigma}{r_0^\sigma},$$

then

$$\sup_{x \in B_{2r}(x_0)} \int_{B_{2r}(x_0)} \frac{V(y)}{|x-y|^{N-4}} dy \le \delta,$$

holds, provided that $0 < r \le \vartheta_0 r_0$, where $\vartheta_0 = \delta/(2cC_1)$. Hence, it suffices to apply those lemmas in order to obtain (5.3). Combining (5.3) with

$$\inf_{B_{r/2}(x_0)} u(x) \ge \frac{c}{|B_r(x_0)|} \int_{B_r(x_0)} u(y) \, dy,$$

corresponding to $u \ge 0$ and $-\Delta u \ge 0$, we obtain for $0 < r \le \vartheta_0 r_0$

$$\sup_{B_{r/2}(x_0)} u(x) \le C \inf_{B_{r/2}(x_0)} u(x). \tag{5.4}$$

Finally, the inequality (5.2) follows from (5.4) by applying a standard covering argument in the annulus $\epsilon \le |x| \le (1+\vartheta)\epsilon$. □

5.1. Universal estimates

The following lemma is due to Mitidieri and Pohozaev [17].

Lemma 5.4. *Let $q > 1$. Assume that u is a weak solution of the problem*

$$\Delta^2 u(x) \ge |u(x)|^q, \quad x \in \Omega \subset \mathbb{R}^N, \tag{5.5}$$

then, there exist $C > 0$ independent of u and $R > 0$ such that

$$\int_{B_R} |u(x)|^q dx \le C R^{\frac{N(q-1)-4q}{q-1}}, \tag{5.6}$$

where $B_R = B_R(x_0) \subset \Omega$.

Proof. Choose $\psi \in C_0^\infty(\mathbb{R}^N)$ radially symmetric such that $supp(\psi) = B_2(0)$, $0 \le \psi \le 1$ and $\psi \equiv 1$ in $B_1(0)$. Given $x_0 \in \Omega$ and $R > 0$ such that $B_{2R}(x_0) \subset\subset \Omega$, define $\psi_R = \psi(R^{-1}(x-x_0))$. Multiply (5.5) by ψ and integrate by parts. By Hölder inequality, we get

$$\int_\Omega |u|^q \psi_R dx \le \int_\Omega \Delta^2 u \psi_R dx \le \int_\Omega u \Delta^2 \psi_R dx \le \left(\int_\Omega |u|^q \psi_R dx\right)^{\frac{1}{q}} \left(\int_\Omega \frac{|\Delta^2\psi|^{q'}}{\psi^{q'-1}} dx\right)^{\frac{1}{q'}}.$$

By a standard change of variables we obtain:

$$\int_{B_R} |u|^q dx \le C \left(\int_{B_1} \frac{|\Delta^2\psi|^{q'}}{\psi^{q'-1}} dx\right)^{\frac{1}{q'}} R^{\frac{N(q-1)-4q}{q-1}}.$$

This concludes the proof. $\square$

Theorem 5.5. *Let $N > 4$ and $\Omega \neq \mathbb{R}^N$. Let $f : \mathbb{R} \to \mathbb{R}$ be a given function such that there exists $q \in (1, N/(N-4))$ and $K > 1$ such that*

$$|u|^q \le f(u) \le K|u|^q, \ u \in \mathbb{R} \tag{5.7}$$

Then, there exist $C_1 > 0$ and $C_2 > 0$ (depending only on N, q and K) such that if $u \in L^q_{\mathrm{loc}}(\Omega)$ is a weak solution of

$$\Delta^2 u(x) = f(u(x)), \quad x \in \Omega, \tag{5.8}$$

then, the following estimates hold

$$|u(x)| \le C_1 \ d(x, \partial\Omega)^{-\frac{4}{q-1}}, \quad x \in \Omega, \tag{5.9}$$

$$|\Delta u(x)| \le C_2 \ d(x, \partial\Omega)^{-\frac{2(q+1)}{q-1}}, \quad x \in \Omega. \tag{5.10}$$

Proof. Let $x_0 \in \Omega$ and $R_0 > 0$ be such that $B_{2R_0}(x_0) \subset\subset \Omega$. Since $q < \frac{N}{N-4}$, we can fix $\sigma > 0$ such that

$$q = \frac{N}{N-4+\sigma} < \frac{N}{N-4}.$$

It follows that $V \equiv |u|^{q-1} \in L^{N/(4-\sigma)}_{\mathrm{loc}}(\Omega)$. In view of Example 2.4 we know that, $V \in K^{N,2}_{\mathrm{loc}}(\Omega)$.

Let $x_0 \in \Omega$ and R_0 be chosen as in Theorem 3.3. Take $\delta_0 = R/8$. From (5.7) and Lemma 2.7, we get:

$$|u(x)| \le C\, R_0^{4-N} \|Vu\|_{L^1(B_{\frac{3R}{2}}(x_0)} + C\, R_0^{-N} \|u\|_{L^1(B_{\frac{R_0}{4}+R}(x_0))}$$

where C and C_1 are positive constants which depend on N. From Lemma 5.4 and Hölder inequality, we get that

$$|u(x)| \leq C_1 R_0^{4-N} R_0^{\frac{N(q-1)-4q}{q-1}} + C_2 R_0^{-N} R_0^{\frac{N(q-1)-4q}{q(q-1)}} R_0^{\frac{N}{q'}}, \tag{5.11}$$

which implies that for any $x \in B_R(x_0)$

$$|u(x)| \leq C\, R^{-\frac{4}{q-1}}.$$

Thus (5.9) follows by choosing $R = d(x, \partial\Omega)$. A similar argument as above can be used to show that indeed (5.10) holds. We omit the details. □

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Clément Interpolation and Its Role in Adaptive Finite Element Error Control

Carsten Carstensen

In honor of the retirement of Philippe Clément.

Abstract. Several approximation operators followed Philippe Clément's seminal paper in 1975 and are hence known as Clément-type interpolation operators, weak-, or quasi-interpolation operators. Those operators map some Sobolev space $V \subset W^{k,p}(\Omega)$ onto some finite element space $V_h \subset W^{k,p}(\Omega)$ and generalize nodal interpolation operators whenever $W^{k,p}(\Omega) \not\subset C^0(\overline{\Omega})$, i.e., when $p \leq n/k$ for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. The original motivation was $H^2 \not\subset C^0(\Omega)$ for higher dimensions $n \geq 4$ and hence nodal interpolation is not well defined.

Todays main use of the approximation operators is for a reliability proof in a posteriori error control. The survey reports on the class of Clément type interpolation operators, its use in a posteriori finite element error control and for coarsening in adaptive mesh design.

1. Introduction: Motivation and applications

The finite element method (FEM) is the driving force and dominating tool behind today's computational sciences and engineering. It is common sense that the FEM should be implemented in an adaptive mesh-refining version with a posteriori error control. This paper high-lights one mathematical key tool in the justification of those adaptive finite element methods (AFEMs) which dates back to Philippe Clément's seminal paper in 1975.

The origin of his operators, today known under the description Clément-type interpolation operators, weak-, or quasi-interpolation operators in the literature, however, was completely different.

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Indeed, many second-order elliptic boundary value problems are recast in a weak form

$$a(u, v) = b(v) \quad \text{for all } v \in V,$$

where (V, a) is some Hilbert space with induced norm $\|\cdot\|_a$ and $b \in V^*$ has the Riesz representation $u \in V$. Given a subspace V_h, the discrete solution $u_h \in V_h$ satisfies

$$a(u_h, v_h) = b(v_h) \quad \text{for all } v_h \in V_h.$$

The error $e := u - u_h \in V$ then satisfies the *best-approximation property*

$$\|e\|_a = \min_{v_h \in V_h} \|u - v_h\|_a.$$

(The proof is given at the end of Subsection 3.1 below for completeness.)

The classical estimation of the upper bound $\|u - v_h\|_a$ replaces v_h by the nodal interpolation operator applied to the exact solution u. In the applications, one typically has

$$u \in V \cap H^s(\Omega; \mathbb{R}^m)$$

for some Sobolev space $H^s(\Omega; \mathbb{R}^m)$ on some domain Ω and some regularity parameter $s \geq 1$. The higher regularity with $s > 1$ is subject to regularity theory of elliptic PDEs and s depends on the smoothness of data and coefficients as well as on the boundary conditions and on the geometry (e.g., corners) of Ω. An optimal regularity for 2D and convex domains or C^2 boundaries is $s = 2$. For smaller s and/or for higher space dimensions, there holds

$$u \in V \cap H^s(\Omega; \mathbb{R}^m) \not\subset C(\overline{\Omega}; \mathbb{R}^m).$$

Thus the nodal interpolation operator, which evaluates the function u at a finite number of points, is *not* defined well.

The Clément operators $J(v)$ cures that difficulty in that they replace the evaluation at discrete points by some initial local best approximation of v in the L^2 sense followed by the point evaluation of this local best approximation at the discrete degrees of freedom.

This small modification is essential in order to receive approximation and stability properties in the sense that

$$\|v - J(v)\|_{H^r(\Omega)} \leq C\, h^{s-r} \|v\|_{H^s(\Omega)}$$

with some mesh-size independent constant $C > 0$ and the maximal mesh-size $h > 0$. The point is that the range for the exponents s and r is possible with $r = 0, 1$ and $s = 1, 2$ while $s = 1$ is excluded for the nodal interpolation operator even for dimension $n \geq 2$.

At the time of Clément's research, this improvement seemed to be regarded as a marginal technicality in the a priori error control – this possible underestimation is displayed in Ciarlet's finite element book [17] where Clément's research is summarized as an exercise (3.2.3).

The relevance of Clément's results was highlighted in the a posteriori error analysis of the last two decades where the aforementioned first-order approximation and stability property of J for $r \leq s = 1$ had been exploited. Amongst the most influential pioneering publications on a posteriori error control are [2, 21, 5, 19] followed by many others. The readers may find it rewarding to study the survey articles of [20, 7] and the books of [26, 1, 3, 4] for a first insight and further references.

This paper gives a very general approach to Clément-type interpolation in Section 2 and comments on its applications in the a posteriori error analysis in Section 3 where further applications such as the error reduction in adaptive finite element schemes and the coarsening are seen as currently open fields.

2. Clément-type approximation

This section aims at a first glance of a class of approximation operators $J : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ with discrete image which are named after Philippe Clément and called Clément-type interpolation operators or sometimes *quasi-interpolation operators*.

The most relevant properties include the local *first order approximation*, i.e., for all $v \in V \subset W^{1,p}(\Omega)$ with weak gradient $Dv \in L^p(\Omega)^n$ in $\Omega \subset \mathbb{R}^n$ and an underlying triangulation with local mesh-size $h \in L^\infty(\Omega)$ and discrete space V_h, there holds

$$\|v - J(v)\|_{L^p(\Omega)} \leq C\,\|hDv\|_{L^p(\Omega)} \quad \text{and} \quad \|h^{-1}(v - J(v))\|_{L^p(\Omega)} \leq C\,\|Dv\|_{L^p(\Omega)},$$

and the $W^{1,p}(\Omega)$ *stability property*, i.e., for $v \in V$,

$$\|DJ(v)\|_{L^p(\Omega)} \leq C\,\|Dv\|_{L^p(\Omega)}.$$

A discussion on a particular operator with an additional *orthogonality property* ends this section. Although the emphasis in this section is not on the evaluation of constants and their sharp estimation, there is a list $M_1, \ldots, M_5$ of highlighted relevant parameters.

Definition 2.1 (abstract assumptions). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, let $1 < p, q < \infty$ with $1/p + 1/q = 1$ and $m, n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Suppose that $(\varphi_z : z \in \mathcal{N})$ is a Lipschitz continuous partition of unity [associated to V_h below] on Ω with*

$$\omega_z := \{x \in \Omega : \varphi(x) \neq 0\} \subseteq \Omega_z \subset \Omega.$$

The sets Ω_z are supposed to be open, nonvoid, and connected supersets of ω_z of volume $|\Omega_z|$. For any $z \in \mathcal{N}$ let $A_z \subseteq \mathbb{R}^m$ be nonvoid, convex, and closed and let $\Pi_z : \mathbb{R}^m \to \mathbb{R}^m$ be the orthogonal projection onto A_z in $\mathbb{R}^m$ (with respect to the Euclidean metric). Let

$$V \subseteq W^{1,p}(\Omega; \mathbb{R}^m) \quad \text{and} \quad V_h := \{\sum_{z \in \mathcal{N}} a_z \varphi_z : a_z \in A_z\}.$$

For any $z \in \mathcal{N}$ *denote* $V_z := V|_{\Omega_z} := \{v|_{\Omega_z} : v \in V\} \subseteq W^{1,p}(\Omega_z;\mathbb{R}^m)$ *and let* $H_z > 0$ *and* $e_z > 0$. *Suppose there exists a map*

$$J_z : V_z \to \mathbb{R}^m$$

such that the following two hypothesis (H1)–(H2) *hold for all* $z \in \mathcal{N}$ *and for all* $v \in V_z$ *with* $\bar{v}_z := |\Omega_z|^{-1} \int_{\Omega_z} v(x)\,dx \in \mathbb{R}^m$:

(H1) $\|v - \Pi_z(\overline{v}_z)\|_{L^p(\Omega_z)} \le H_z \|Dv\|_{L^p(\Omega_z)}$;
(H2) $\|J_z(v|_{\Omega_z}) - \bar{v}_z\|_{L^p(\Omega_z)} \le e_z H_z \|Dv\|_{L^p(\Omega_z)}$.

Typically, (H1) describes some Poincaré-Friedrichs inequality with $H_z \lesssim \operatorname{diam}(\Omega_z)$ where Ω_z is some local neighborhood of a node.

The subsequent list of parameters illustrates constants crucial in the analysis of the approximation and stability estimates.

Definition 2.2 (Some constants). *Under the assumptions of the previous definition set* (*the piecewise constant function*)

$$H(x) := \max_{\substack{z\in\mathcal{N}\\ x\in\Omega_z}} H_z \quad \textit{for } x \in \Omega$$

and, with $1/p + 1/q = 1$, *define* $M_1, \dots, M_5$ *by*

$$\begin{aligned}
M_1 &:= \max_{z\in\mathcal{N}} \|\varphi_z\|_{L^\infty(\Omega)},\\
M_2 &:= \max_{x\in\overline{\Omega}} \sum_{z\in\mathcal{N}} |\varphi_z(x)|,\\
M_3 &:= \max_{x\in\Omega} \operatorname{card}\{z \in \mathcal{N} : x \in \Omega_z\},\\
M_4 &:= \max_{z\in\mathcal{N}} (1 + e_z),\\
M_5 &:= \sup_{x\in\Omega} \left(\sum_{z\in\mathcal{N}} H(x)^q |D\varphi_z(x)|^q \right)^{1/q}.
\end{aligned}$$

Example 2.3 (P_1FEM in 2D). Given a regular triangulation $\mathcal{T}$ of the unit square $\Omega = (0,1)^2$ into triangles, $m = 1, n = 2$, let $V := H^1_0(\Omega)$, $p = q = 2$, and let $(\varphi_z : z \in \mathcal{N})$ denote the nodal basis functions of all the nodes with respect to the first-order finite element space. The boundary conditions are described by the sets $(A_z : z \in \mathcal{N})$. For each node z in the interior, written $z \in \mathcal{N} \cap \Omega$, $A_z = \mathbb{R}$, while $A_z := \{0\}$ for z on the boundary, written $z \in \mathcal{N} \cap \partial\Omega$, $\Gamma_D = \partial\Omega$. Let J_z denote the integral mean operator

$$J_z(v) := |\omega_z|^{-1} \int_{\omega_z} v(x)\,dx \quad \text{for } v \in V_z \subseteq L^p(\Omega_z)$$

on the patch $\omega_z = \Omega_z = \{x \in \Omega : \varphi(x) > 0\}$ with $e_z = 0$ in (H2). The Dirichlet boundary conditions are then prescribed by ($v \in \mathbb{R}^m$)

$$\Pi_z(v) := \begin{cases} v & \text{if } z \in \mathcal{N} \cap \Omega,\\ 0 & \text{if } z \in \mathcal{N} \cap \partial\Omega. \end{cases}$$

There holds (H1) with Poincaré and Friedrich's inequalities and

$$H_z := \begin{cases} c_P(\omega_z) & \text{if } z \in \mathcal{N} \cap \Omega, \\ c_F(\omega_z, (\partial\Omega) \cap (\partial\omega_z)) & \text{if } z \in \mathcal{N} \cap \partial\Omega; \end{cases}$$

H_z is of the form global constant times $\operatorname{diam}(\omega_z)$. It is always true that

$$M_1 = M_2 = 1 = M_4 \quad \text{and} \quad M_3 = 3.$$

Since $\|D\varphi_z\|_{L^\infty(\Omega)} \approx H_z^{-1}$ there holds

$$M_5 \lesssim 1.$$

Notice that M_5 may be very large for small angles in the triangulation while the constants $M_1, \ldots, M_4$ are robust with respect to small aspect ratios.

The aforementioned example is not exactly the choice of [18] but certainly amongst the natural choices [14, 25].

The announced approximation and stability properties are provided by the following main theorem.

Theorem 2.4. *The map* $J : V \to V_h; v \mapsto \sum_{z\in\mathcal{N}} \Pi_z(J_z(v|_{\Omega_z}))\varphi_z$ *satisfies*

(a) $\exists c_1 > 0\, \forall v \in V$, $\|v - J(v)\|_{L^p(\Omega)} \le c_1\|HDv\|_{L^p(\Omega)}$;
(b) $\exists c_2 > 0\, \forall v \in V$, $\|H^{-1}(v - J(v))\|_{L^p(\Omega)} \le c_2\|Dv\|_{L^p(\Omega)}$,
(c) $\exists c_3 > 0\, \forall v \in V$, $\|DJ(v)\|_{L^p(\Omega)} \le c_3\|Dv\|_{L^p(\Omega)}$.

The constants c_1 *and* c_2 *depend on* $M_1, \ldots, M_4$ *and on* p, q, m, n *while* c_3 *depends also on* M_5.

Proof. Given $v \in V$, let $v_z := J_z(v|_{\Omega_z})$, $z \in \mathcal{N}$, be the coefficient in

$$J(v) = \sum_{z\in\mathcal{N}} \Pi_z(v_z)\varphi_z$$

and let $\overline{v}_z := |\Omega_z|^{-1} \int_{\Omega_z} v(x)\, dx$ be the local integral mean.

The first step of this proof establishes the estimate

$$\|v - \Pi_z(v_z)\|_{L^p(\Omega_z)} \le M_4 H_z \|Dv\|_{L^p(\Omega_z)}. \tag{2.1}$$

The assumptions (H1) and (H2) lead to

$$\|v|_{\Omega_z} - \Pi_z(\overline{v}_z)\|_{L^p(\Omega_z)} \le H_z\|Dv\|_{L^p(\Omega_z)},$$
$$\|\overline{v}_z - v_z\|_{L^p(\Omega_z)} \le e_z H_z \|Dv\|_{L^p(\Omega_z)}.$$

Since Π_z is non-expansive (Lipschitz with $\operatorname{Lip}(\Pi_z) \le 1$; $\mathbb{R}^m$ is endowed with the Euclidean norm used in $L^p(\Omega; \mathbb{R}^m)$) there holds

$$\|\Pi_z(\overline{v}_z) - \Pi_z(v_z)\|_{L^p(\Omega_z)} \le \|\overline{v}_z - v_z\|_{L^p(\Omega)}.$$

The combination of the aforementioned estimates yields an upper bound of the right-hand side in

$$\|v - \Pi_z(v_z)\|_{L^p(\Omega_z)} \le \|v - \Pi_z(\overline{v}_z)\|_{L^p(\Omega_z)} + \|\Pi_z(\overline{v}_z) - \Pi(v_z)\|_{L^p(\Omega_z)}$$

and, in this way, establishes (2.1) and concludes the first step.

Step two establishes assertion (a) of the theorem. Since $(\varphi_z)_{z\in\mathcal{N}}$ is a partition of unity, $\sum_{z\in\mathcal{N}}\varphi_z = 1$ in Ω, there holds

$$\begin{aligned}\|v - J(v)\|^p_{L^p(\Omega)} &= \|\sum_{z\in\mathcal{N}}(v - \Pi_z(v_z))\varphi_z\|^p_{L^p(\Omega)}\\ &= \int_\Omega \left|\sum_{z\in\mathcal{N}} \varphi_z^{1/q}\varphi_z^{1/p}(v - \Pi_z(v_z))\right|^p dx.\end{aligned}$$

Hölders's inequality in ℓ^1 and $\sum_{z\in\mathcal{N}}|\varphi_z| \le M_2$ lead to

$$\begin{aligned}\|v - J(v)\|^p_{L^p(\Omega)} &\le \int_\Omega \left(\sum_{z\in\mathcal{N}}|\varphi_z|\,|v - \Pi_z(v_z)|^p\right)\left(\sum_{z\in\mathcal{N}}|\varphi_z|\right)^{p/q} dx\\ &\le M_2^{p/q}\sum_{z\in\mathcal{N}}\int_\Omega |\varphi_z|\,|v - \Pi_z(v_z)|^p\,dx\\ &\le M_1 M_2^{p/q}\sum_{z\in\mathcal{N}}\|v - \Pi_z(v_z)\|^p_{L^p(\Omega_z)}.\end{aligned}$$

This and estimate (2.1) from the first step show assertion (a) with

$$c_1 := M_1^{1/p} M_2^{1/q} M_3^{1/p} M_4.$$

In fact, the last step involves the overlaps by means of M_3 and the definition of $H(x) := \max\{H_z : \exists z \in \mathcal{N}, x \in \Omega_z\}$:

$$\begin{aligned}\sum_{z\in\mathcal{N}} H_z^p\|Dv\|^p_{L^p(\Omega_z)} &\le \int_\Omega \sum_{\substack{z\in\mathcal{N}\\ x\in\Omega_z}} H_z^p|Dv(x)|^p\,dx\\ &\le \int_\Omega H(x)^p(\sum_{\substack{z\in\mathcal{N}\\ x\in\Omega_z}} 1)|Dv(x)|^p\,dx\\ &\le M_3\|H\,Dv\|^p_{L^p(\Omega)}.\end{aligned}$$

This concludes step two and the proof of assertion (a).

Step three establishes assumption (b) of the theorem. The same list of arguments as in step two shows

$$\|H^{-1}(v - J(v))\|^p_{L^p(\Omega)} \le M_1 M_2^{p/q}\sum_{z\in\mathcal{N}}\|H^{-1}(v - \Pi_z(v_z))\|^p_{L^p(\Omega_z)}.$$

Since $H_z \le H(x)$ for $x \in \Omega_z$, $z \in \mathcal{N}$, there holds

$$\|H^{-1}(v - J(v))\|^p_{L^p(\Omega)} \le M_1 M_2^{p/q}\sum_{z\in\mathcal{N}} H_z^{-p}\|v - \Pi_z(v_z)\|^p_{L^p(\Omega_z)}.$$

Based on (2.1), the proof of assertion (b) and step three are concluded as in step two. This yields (b) with $c_2 := M^{1/p} M_2^{1/q} M_3^{1/p} M_4$.

Step four establishes assumption (c) of the theorem. Since

$$\sum_{z\in\mathcal{N}} \varphi_z = 1 \text{ there holds } \sum_{z\in\mathcal{N}} D\varphi_z = 0$$

almost everywhere in Ω. With the characteristic function $\chi_z \in L^\infty(\Omega)$ of Ω_z, defined by $\chi_z|_{\Omega_z} = 1$ and $\chi_z|_{\Omega\setminus\Omega_z} = 0$ it follows that

$$\|DJ(v)\|^p_{L^p(\Omega)} = \int_\Omega \left| \sum_{z\in\mathcal{N}} \chi_z(x) H^{-1}(x)(\Pi_z(v_z) - v(x)) D\varphi_z(x) H(x) \right|^p dx.$$

Hölder's inequality in ℓ^1 shows, for almost every $x \in \Omega$,

$$\begin{aligned}
&\left| \sum_{z\in\mathcal{N}} \chi_z(x) H^{-1}(x)(\Pi_z(v_z) - v(x)) D\varphi_z(x) H(x) \right| \\
&\le \left(\sum_{z\in\mathcal{N}} \chi_z(x) H^{-p}(x) |\Pi_z(v_z) - v(x)|^p \right)^{1/p} \left(\sum_{z\in\mathcal{N}} H(x)^q |D\varphi_z(x)|^q \right)^{1/q} \\
&\le M_5 \left(\sum_{z\in\mathcal{N}} \chi_z(x) H_z^{-p} |\Pi_z(v_z) - v(x)|^p \right)^{1/p}.
\end{aligned}$$

The combination of the two preceding estimates in this step four is followed by (2.1) and then results in

$$\|DJ(v)\|^p_{L^p(\Omega)} \le M_4^p M_5^p \sum_{z\in\mathcal{N}} \|Dv\|^p_{L^p(\Omega_z)}.$$

The proof concludes as in step two and establishes assertion (c) with $c_3 := M_3^{1/p} M_4 M_5$. □

In order to discuss a particular operator designed to introduce an extra orthogonality property, additional assumptions are necessary.

Definition 2.5 (Free nodes for Dirichlet boundary conditions). *Adopt the notation of Definition* 2.1 *and suppose, in addition, that*

$$V_h \subset V := \{v \in W^{1,p}(\Omega; \mathbb{R}^m) : v = 0 \text{ on } \Gamma_D\},$$

where Γ_D *is some (possibly empty) closed part of the boundary that includes a complete set of edges in the sense that, for each* $E \in \mathcal{E}$, *either* $E \subset \Gamma_D$ *or* $E \cap \Gamma_D \subset \mathcal{N}$. *Let* $\mathcal{K} := \mathcal{N} \setminus \Gamma_D$ *and, for all* $z \in \mathcal{N}$,

$$A_z = \begin{cases} \mathbb{R}^m & \text{if } z \in \mathcal{K}; \\ \{0\} & \text{otherwise.} \end{cases}$$

Given any $z \in \mathcal{K}$ *let* $\mathcal{J}(z) \subset \mathcal{N}$ *such that* $z \in \mathcal{J}(z)$ *and*

$$\psi_z := \sum_{x\in\mathcal{J}(z)} \varphi_x$$

defines a Lipschitz partition $(\psi_z : z \in \mathcal{K})$ *of unity with* $\Omega_z \supseteq \{x \in \Omega : \psi_z(x) \neq 0\}$. *Then*

$$J_z(v) := \frac{\int_{\Omega_z} v\psi_z\,dx}{\int_{\Omega_z} \varphi_z\,dx}$$

defines an approximation operator $J : V \to V_h$. *Set*

$$M_6 := \max_{z\in\mathcal{K}} \|\varphi_z\|^{-1}_{L^1(\Omega_z)} \|\psi_z\|_{L^q(\Omega_z)} \|w\|_{L^p(\Omega_z)} |\Omega_z|^{1/p}.$$

Theorem 2.6. *The operator* $J : V \to V_h$ *from Definition* 2.5 *satisfies* (a)–(c) *from Theorem* 2.4 *plus the* orthogonality property

$$\left|\int_\Omega f\cdot(v - Jv)\,dx\right| \le c_4 \|Dv\|_{L^p(\Omega)} \left(\sum_{z\in\mathcal{K}} H_z^q \min_{f_z\in\mathbb{R}^m} \|f - f_z\|^q_{L^q(\Omega_z)}\right)^{1/q}.$$

Remark 2.7. A Poincaré inequality in case $f \in W^{1,q}(\Omega)$,

$$\|f - f_z\|_{L^q(\Omega_z)} \le C_p(\Omega_z) H_z \|Df\|_{L^q(\Omega_z)},$$

illustrates that the right-hand side in Theorem 2.6 is of the form

$$O(\|H\|^2_{L^\infty(\Omega)})\|Dv\|_{L^p(\Omega)}$$

and then of higher order (compared with the error of first-order FEM).

Proof of Theorem 2.6. In order to employ Theorem 2.4, it remains to check the conditions (H1)–(H2). For any $v \in V_z = V|_{\Omega_z}$, (H1) is either a Poincaré or Friedrichs inequality for $z \in \mathcal{K}$ or $z \in \mathcal{K} \cap \Gamma_D$ as in Example 2.3. Moreover, given any $v \in V_z$ and

$$J_z(v) := \int_\Omega v\psi_z\,dx \Big/ \int_\Omega \varphi_z\,dx \quad \in \mathbb{R}^m,$$

there holds $J_z(\overline{v}_z) = \overline{v}_z$ if $\psi_z \equiv \varphi_z$ (i.e., $\mathcal{J}(z) = \{z\}$) for $\overline{v}_z := |\Omega_z|^{-1} \int_{\Omega_z} v(x)\,dx$. Then, for $w = v|_{\Omega_z} - \overline{v}_z$, there holds

$$\begin{aligned} \|J_z(v|_{\Omega_z}) - \overline{v}_z\|_{L^p(\Omega_z)} &= \|J_z(w)\|_{L^p(\Omega_z)} \\ &\le \|\varphi_z\|^{-1}_{L^1(\Omega_z)} \|\psi_z\|_{L^q(\Omega_z)} \|w\|_{L^p(\Omega_z)} |\Omega_z|^{1/p} \\ &\le M_6 \|w\|_{L^p(\Omega_z)} \le M_6 H_z \|Dv\|_{L^p(\Omega_z)}. \end{aligned}$$

For $\psi_z \not\equiv \varphi_z$ (i.e., $\mathcal{J}(z) \neq \{z\}$), $\overline{\Omega}_z$ includes nodes at the boundary and $\partial\Omega_z \cap \Gamma_D$ has positive surface measure. Therefore, Friedrichs inequality guarantees

$$\|v_z\|_{L^p(\Omega_z)} \le c_5 H_z \|Dv_z\|_{L^p(\Omega_z)} \quad \text{for all } v_z \in V_z.$$

Then (H2) follows immediately from this and

$$\|J_z(v)\|_{L^p(\Omega_z)} + \|\overline{v}_z\|_{L^p(\Omega_z)} \le c_6 \|v_z\|_{L^p(\Omega_z)}.$$

The orthogonality property is based on the design of $J_z(v|_{\Omega_z})$ and the partition of unity $(\psi_z : z \in \mathcal{K})$. In fact,

$$\int_{\Omega_z} \left(v(x)\psi_z(x) - J_z(v|_{\Omega_z})\varphi_z(x)\right)\,dx = 0$$

for any $z \in \mathcal{K}$ and so, for any $f_z \in \mathbb{R}^m$, there follows

$$\begin{aligned}
&\int_\Omega f(v - Jv)\,dx \\
&\quad = \int_\Omega f(x)\left(\sum_{z\in\mathcal{K}} v(x)\psi_z(x) - \sum_{z\in\mathcal{K}} J_z(v|_{\Omega_z})\varphi_z(x)\right)\,dx \\
&\quad = \sum_{z\in\mathcal{K}} \int_{\Omega_z} f(x)\cdot\left(v(x)\psi_z(x) - J_z(v|_{\Omega_z})\varphi_z(x)\right)\,dx \\
&\quad = \sum_{z\in\mathcal{K}} \int_{\Omega_z} (f(x) - f_z)\cdot\left(v(x)\psi_z(x) - J_z(v|_{\Omega_z})\varphi_z(x)\right)dx \\
&\quad \le \sum_{z\in\mathcal{K}} \left(H_z\|f - f_z\|_{L^q(\Omega_z)}\right)\left(H_z^{-1}\|v\psi_z - J_z(v|_{\Omega_z})\varphi_z\|_{L^p(\Omega_z)}\right) \\
&\quad \le \left(\sum_{z\in\mathcal{K}} H_z^q\|f - f_z\|^q_{L^q(\Omega_z)}\right)^{1/q}\left(\sum_{z\in\mathcal{K}} H_z^{-p}\|v\psi_z - J_z(v|_{\Omega_z})\varphi_z\|^p_{L^p(\Omega_z)}\right)^{1/p}.
\end{aligned}$$

For $\psi_z \equiv \varphi_z$ (i.e., $\mathcal{J}(z) = \{z\}$) it follows

$$\|v(\psi_z - \varphi_z)\|_{L^p(\Omega_z)} + \|(v - J_z(v|_{\Omega_z}))\varphi_z\|_{L^p(\Omega_z)} \le c_7 H_z\|Dv\|_{L^p(\Omega_z)}$$

as above. For $\psi_z \not\equiv \varphi_z$, a Friedrichs inequality shows

$$\|v\|_{L^p(\Omega)} \le c_8 H_z\|Dv\|_{L^p(\Omega_z)}$$

and completes the proof. The remaining details are similar to the arguments of Theorem 2.4 and hence omitted. □

3. A posteriori residual control

This section reviews explicit residual-based error estimators in a very abstract form. The subsequent benchmark example motivates this analysis of the dual norm of a linear functional.

3.1. Model example

For the benchmark example suppose that an underlying PDE yields to some bilinear form a on some Sobolev space V such that (V, a) is Hilbert space and the right-hand side b is a bounded linear functional, written $b \in V^*$. The Riesz representation u of b is called weak solution and is characterized by $u \in V$ with

$$a(u, v) = b(v) \quad \text{for all } v \in V.$$

The finite element solution u_h belongs to some finite-dimensional subspace V_h of V, the finite element space, and is characterized by $u_h \in V_h$ with

$$a(u_h, v_h) = b(v_h) \quad \text{for all } v_h \in V_h.$$

In other words, u_h is the Riesz representation of the restricted right-hand side $b|_{V_h}$ in the discrete Hilbert space $(V_h, a|_{V_h\times V_h})$.

The goal of a posteriori error control is to represent the error

$$e := u - u_h \in V$$

in the energy norm $\|\cdot\|_a = \sqrt{a(\cdot,\cdot)}$. One particular property is the Galerkin orthogonality

$$a(e, v_h) = 0 \quad \text{for all } v_h \in V_h.$$

This followed by a Cauchy inequality yields, for all $v_h \in V_h$, that

$$\|e\|_a^2 = a(e, u - v_h) \le \|e\|_a \, \|u - v_h\|_a.$$

This implies the best-approximation property

$$\|e\|_a = \min_{v_h \in V_h} \|u - v_h\|_a$$

claimed in the introduction.

3.2. Error and residual

Given a, b, and u_h, the residual

$$R := b - a(u_h, \cdot) \in V^*$$

is a bounded linear functional in V, written $R \in V^*$. Its dual norm is

$$\|R\|_{V^*} := \sup_{v \in V \setminus \{0\}} R(v)/\|v\|_a = \sup_{v \in V \setminus \{0\}} a(e, v)/\|v\|_a = \|e\|_a < \infty. \tag{3.1}$$

The first equality in (3.1) is a definition, the second equality immediately follows from

$$R = a(e, \cdot).$$

Notice that the Galerkin orthogonality immediately translates into

$$V_h \subset \text{kern } R.$$

A Cauchy inequality with respect to the scalar product a results in

$$\|R\|_{V^*} \le \|e\|_a$$

and hence in one of two inequalities of the last identity $\|R\|_{V^*} = \|e\|_a$ in (3.1). The remaining converse inequalities follows in fact with $v = e$ yields finally the equality; the proof of (3.1) is finished.

The identity (3.1) means that the error (estimation) in the energy norm *is equivalent* to the (computation of the) dual norm of the given residual. Furthermore, it is even of *comparable computational effort* to compute an optimal $v = e$ in (3.1) or to compute e. The proof of (3.1) yields even a stability estimate: Given any $v \in V$ with $\|v\|_a = 1$, the relative error of $R(v)$ as an approximation to $\|e\|_a$ equals

$$\frac{\|e\|_a - R(v)}{\|e\|_a} = 1/2 \, \| \, v - e/\|e\|_a \, \|_a^2. \tag{3.2}$$

In fact, given any $v \in V$ with $\|v\|_a = 1$, the identity (3.2) follows from

$$1 - a(e/\|e\|_a, v) = \tfrac{1}{2}\, a(e/\|e\|_a, e/\|e\|_a) - a(e/\|e\|_a, v) + \tfrac{1}{2}\, a(v, v)$$
$$= \tfrac{1}{2}\, \|v - e/\|e\|_a\|_a^2. \qquad \square$$

The interpretation of the error estimate (3.2) is as follows. The maximizing v in (3.1) (i.e., $v \in V$ with maximal $R(v)$ subject to $\|v\|_a \leq 1$) is unique and equals $e/\|e\|_a$. As a consequence, the computation of the maximizing v in (3.1) is equivalent to and indeed equally expensive as the computation of the unknown $e/\|e\|_a$ and so essentially of the exact solution u.

Therefore, a posteriori error analysis aims to compute lower and upper bounds of $\|R\|_{V^*}$ instead of its exact value. However, the use of any good guess of the exact solution from some postprocessing with extrapolation or superconvergence phenomenon is welcome (and, e.g., may enter v).

3.3. Residual representation formula

Without stating examples in explicit form, it is stressed that linear and nonlinear boundary value problems of second order elliptic PDEs typically lead (after some partial integration) to some explicit representation of the residual $R := b - a(u_h, \cdot)$. Let $\mathcal{T}$ denote the underlying regular triangulation of the domain and let $\mathcal{E}$ denote a set of edges for $n = 2$. Then, the data and the discrete solution u_h lead to given quantities $r_T \in \mathbb{R}^m$, the explicit volume residual, and $r_E \in \mathbb{R}^m$, the jump residual, for any element domain T and any edge E such that the *residual representation formula*

$$R(v) = \sum_{T \in \mathcal{T}} \int_T r_T \cdot v\, dx - \sum_{E \in \mathcal{E}} \int_E r_E \cdot v\, ds \tag{3.3}$$

holds for all $v \in V$.

The goal of a posteriori error control is to establish guaranteed lower and upper bounds of the (energy norm of the) error and hence of the (dual norm of the) residual.

Any choice of $v \in V \setminus \{0\}$ immediately leads to a lower error bound via

$$\eta := |R(v)/\|v\|_a| \leq \|e\|_a.$$

Indeed, edge and element-oriented bubble functions are studied for v in [26] to prove efficiency.

More challenging appears reliability, i.e., the design of upper error bounds which require extra conditions. In fact, the Galerkin orthogonality $V_h \subset \operatorname{kern} R$ is supposed to hold for the data in (3.3).

3.4. Explicit residual-based error estimators

Given the explicit volume and jump residuals r_T and r_E in (3.3), one defines the *explicit residual-based estimator*

$$\eta_R^2 := \sum_{T \in \mathcal{T}} h_T^2\, \|r_T\|_{L^2(T)}^2 + \sum_{E \in \mathcal{E}_\Omega} h_E\, \|r_E\|_{L^2(E)}^2, \tag{3.4}$$

which is reliable in the sense that

$$\|e\|_a \le C\,\eta_R. \tag{3.5}$$

The proof of (3.5) follows from the Clément-type interpolation operators and their properties. In fact, given any $v \in V$ and any $v_h \in V_h$ set $w := v - v_h$. Then the residual representation formula with $V_h \subset \operatorname{kern} R$ plus Cauchy inequalities lead to

$$\begin{aligned}
R(v) &= R(w) = \sum_{T\in\mathcal{T}} \int_T r_T \cdot w\,dx - \sum_{E\in\mathcal{E}_\Omega} \int_E r_E \cdot w\,ds \\
&\le \sum_{T\in\mathcal{T}} \left(h_T\,\|r_T\|_{L^2(T)}\right)\left(h_T^{-1}\,\|w\|_{L^2(T)}\right) \\
&\qquad + \sum_{E\in\mathcal{E}_\Omega} \left(h_E^{1/2}\,\|r_E\|_{L^2(E)}\right)\left(h_E^{-1/2}\,\|w\|_{L^2(E)}\right) \\
&\le \left(\sum_{T\in\mathcal{T}} h_T^2\,\|r_T\|_{L^2(T)}^2\right)^{1/2} \left(\sum_{T\in\mathcal{T}} h_T^{-2}\,\|w\|_{L^2(T)}^2\right)^{1/2} \\
&\qquad + \left(\sum_{E\in\mathcal{E}_\Omega} h_E\,\|r_E\|_{L^2(E)}^2\right)^{1/2} \left(\sum_{E\in\mathcal{E}_\Omega} h_E^{-1}\,\|w\|_{L^2(E)}^2\right)^{1/2}.
\end{aligned}$$

The well-established trace inequality for each element T of diameter h_T and its boundary ∂T shows

$$h_T^{-1}\|w\|_{L^2(\partial T)}^2 \le C_T\left(h_T^{-2}\|w\|_{L^2(T)}^2 + \|Dw\|_{L^2(T)}^2\right)$$

with some size-independent constant C_T (which solely depends on the shape of the element domain). This allows an estimate of

$$\left(\sum_{E\in\mathcal{E}_\Omega} h_E^{-1}\,\|w\|_{L^2(E)}^2\right)^{1/2} \le C\sum_{T\in\mathcal{T}}\left(h_T^{-2}\,\|w\|_{L^2(T)}^2 + \|Dw\|_{L^2(T)}^2\right).$$

The localized first-order approximation and stability property of the Clément-interpolation shows

$$\sum_{T\in\mathcal{T}}\left(h_T^{-2}\,\|w\|_{L^2(T)}^2 + \|Dw\|_{L^2(T)}^2\right) \le C\sum_{T\in\mathcal{T}} \|Dv\|_{L^2(T)}^2.$$

Altogether, it follows that

$$R(v) \le C\eta_R\,|v|_{H^1(\Omega)}.$$

This proves reliability in the sense of (3.5) □

3.5. Error reduction and discrete residual control

Different norms of the residual play an important role in the design of convergent adaptive meshes. Suppose $V_h \equiv V_\ell$ is the finite element space with a discrete solution $u_h \equiv u_\ell$ and its residual $R \equiv R_\ell := b - a(u_\ell, \cdot)$. The design task in adaptive finite element methods is to find a superspace $V_{\ell+1} \supset V_\ell$ associated with some finer mesh such that the discrete solution $u_{\ell+1}$ in $V_{\ell+1} \subset V$ is strictly more accurate. To make this precise suppose that $0 < \varrho < 1$ and $0 \leq \delta$ satisfy *error reduction* in the sense that

$$\|u - u_{\ell+1}\|_a^2 \leq \varrho \, \|u - u_\ell\|_a^2 + \delta. \tag{3.6}$$

The interpretation is that the error is reduced by a factor at most $\varrho^{1/2} < 1$ up to the terms δ which describe small effects. The error reduction leads to discuss the equivalent *discrete residual control*

$$(1 - \varrho) \, \|R\|_{V^*}^2 \leq \|R\|_{V_{\ell+1}^*}^2 + \delta. \tag{3.7}$$

Therein, $\|R\|_{V_{\ell+1}^*}$ denotes the dual norm of the restricted functional $R|_{V_{\ell+1}}$, namely

$$\|R\|_{V_{\ell+1}^*} := \sup_{v \in V_{\ell+1} \setminus \{0\}} R(v)/\|v\|_a.$$

The point is that (3.6)$\Leftrightarrow$(3.7). The proof follows from (3.1) which leads to

$$\|R\|_{V^*} = \|u - u_\ell\|_a \quad \text{and} \quad \|R\|_{V_{\ell+1}^*} = \|u_{\ell+1} - u_\ell\|_a.$$

This plus the Galerkin orthogonality and the Pythagoras theorem

$$\|u - u_\ell\|_a^2 = \|u - u_{\ell+1}\|_a^2 + \|u_{\ell+1} - u_\ell\|_a^2$$

show that (3.7) reads

$$(1 - \varrho) \, \|u - u_\ell\|_a^2 \leq \|u - u_\ell\|_a^2 - \|u - u_{\ell+1}\|_a^2 + \delta$$

and this is (3.6). Altogether, (3.6)$\Leftrightarrow$(3.7) and one needs to study the *discrete residual control*.

One standard technique is the bulk criterion for the explicit error estimator η_R followed by local discrete efficiency. The latter involves proper mesh-refinement rules and hence this is not included in this paper.

3.6. Remarks

This subsection lists a few short comments on various tasks in the a posteriori error analysis.

3.6.1. Dominating edge contributions. For first-order finite element methods in simple situations of the Laplace, Lamé problem, or the primal formulation of elastoplasticity, the volume term $r_T = f$ can be substituted by the higher-order term of oscillations, i.e.,

$$\|e\|_a^2 \leq C \left(\operatorname{osc}(f)^2 + \sum_{E \in \mathcal{E}_\Omega} h_E \, \|r_E\|_{L^2(E)}^2 \right). \tag{3.8}$$

Therein, for each node $z \in \mathcal{N}$ with nodal basis function φ_z and patch $\omega_z := \{x \in \Omega : \varphi_z(x) \neq 0\}$ of diameter h_z and the source term $f \in L^2(\Omega)^m$ with integral mean $f_z := |\omega_z|^{-1} \int_{\omega_z} f(x) \, dx \in \mathbb{R}^m$ the oscillations of f are defined by

$$\operatorname{osc}(f) := \Big(\sum_{z \in \mathcal{N}} h_z^2 \, \|f - f_z\|_{L^2(\omega_z)}^2 \Big)^{1/2}.$$

Notice for $f \in H^1(\Omega)^m$ and the mesh-size $h_{\mathcal{T}} \in P_0(\mathcal{T})$ there holds

$$\operatorname{osc}(f) \leq C\, \|h_{\mathcal{T}}^2\, Df\|_{L^2(\Omega)}$$

and so $\operatorname{osc}(f)$ is of quadratic and hence of higher order. We refer to [10, 9, 16], [22], [6], and [24] for further details on and proofs of (3.8). The proof in [16] is based on Theorem 2.6. In fact, the arguments of Subsection 3.4 are combined with $r_T = f$ for the lowest-order finite element method at hand. Then, Theorem 2.6 shows

$$\int_\Omega f w\, dx \leq c_4 \|Dv\|_{L^2(\Omega)} \operatorname{osc}(f)$$

and leads to the announced higher-order term.

Dominating edge contributions lead to efficiency of averaging schemes [10, 11].

3.6.2. Reliability constants. The global constants in the approximation and stability properties of Clément-type operators enter the reliability constants. The estimates of [14] illustrate some numbers as local eigenvalue problems. Based on numerical calculations, there is proof that, for all meshes with right-isosceles triangles and the Laplace equation with right-hand side $f \in L^2(\Omega)$, there holds reliability in the sense of

$$\|D(u-u_h)\|_{L^2(\Omega)} \leq \left(\sum_{T\in\mathcal{T}} h_T^2 \|f\|_{L^2(T)}^2\right)^{1/2} + \left(\sum_{E\in\mathcal{E}} h_E \int_E [\partial u_h/\partial \nu_E]^2\, ds\right)^{1/2}$$

with an undisplayed constant $\mathrm{C}_{\mathrm{rel}} < 1$ in the upper bound [15].

The aspect of upper bounds with constants and corresponding overestimation is empirically discussed in [13].

3.6.3. Coarsening. One actual challenging detail is coarsening. Reason number one is that coarsening cannot be avoided for time evolving problems to capture moving singularities. Reason number two is the design of algorithms which are guaranteed to convergence in optimal complexity [8].

Any coarsening excludes the Pythagoras theorem (which assumes $V_\ell \subset V_{\ell+1}$) and so perturbations need to be controlled which leads to the estimation of

$$\min_{v_{\ell+1}\in V_{\ell+1}} \|v_{\ell+1} - u_\ell\|_a.$$

Ongoing research investigates the effective estimation of this term.

3.6.4. Other norms and goal functionals. The previous sections concern estimations of the error in the energy norm. Other norms are certainly of some interest as well as the error with respect to a certain goal functional. The later is some given bounded and linear functional $\ell : V \to \mathbb{R}$ with respect to which one aims to monitor the error. That is, one wants to find computable lower and upper bounds for the (unknown) quantity

$$|\ell(u) - \ell(u_h)| = |\ell(e)|.$$

Typical examples of goal functionals are described by L^2 functions, e.g.,

$$\ell(v) = \int_\Omega \varrho\, v\, dx \quad \forall v \in V$$

for a given $\varrho \in L^2(\Omega)$ or as contour integrals. To bound or approximate $J(e)$ one considers the *dual problem*

$$a(v, z) = \ell(v) \quad \forall v \in V \tag{3.9}$$

with exact solution $z \in V$ (guaranteed by the Lax-Milgram lemma) and the discrete solution $z_h \in V_h$ of

$$a(v_h, z_h) = \ell(v_h) \quad \forall v_h \in V_h.$$

Set $f := z - z_h$. Based on the Galerkin orthogonality $a(e, z_h) = 0$ one infers

$$\ell(e) = a(e, z) = a(e, z - z_h) = a(e, f). \tag{3.10}$$

Cauchy inequalities lead to the a posteriori estimate

$$|\ell(e)| \le \|e\|_a \, \|f\|_a \le \eta_u \eta_z.$$

Indeed, utilizing the primal and dual residual R_u and R_z in V^*, defined by

$$R_u := b - a(u_h, \cdot) \quad \text{and} \quad R_z := \ell - a(\cdot, z_h),$$

computable upper error bounds for $\|e\|_V \le \eta_u$ and $\|f\|_V \le \eta_z$ can be found by the arguments of the energy error estimators [1, 3]. Indeed, the parallelogram rule shows

$$2\,\ell(e) = 2\,a(e, f) = \|e + f\|_a^2 - \|e\|_a^2 - \|f\|_a^2.$$

This right-hand side can be written in terms of residuals, in the spirit of (3.1), namely $\|e\|_a = \|R_u\|_{V^*}$, $\|f\|_a = \|R_z\|_{V^*}$, and

$$\|e + f\|_a = \|R_{u+z}\|_{V^*}$$

for

$$R_{u+z} := b + \ell - a(u_h + z_h, \cdot) = R_u + R_z \in V^*.$$

Therefore, the estimation of $\ell(e)$ is reduced to the computation of lower and upper error bounds for the three residuals R_u, R_z, and R_{u+z} with respect to the energy norm. This illustrates that the energy error estimation techniques of the previous sections may be employed for goal-oriented error control [1, 3].

Alternatively, Rannacher et al. investigated the weighted residual technique where the Clément interpolation operator is employed as well. The reader is referred to [4].

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Ergodic Properties of Reaction-diffusion Equations Perturbed by a Degenerate Multiplicative Noise

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A Philippe con stima e affetto

Abstract. We extend to a more general class of diffusion coefficients the results proved in the previous work [6] on uniqueness, ergodicity and strongly mixing property of the invariant measure for some stochastic reaction-diffusion equations, in which the diffusion term is possibly vanishing and the deterministic part is not asymptotically stable. We obtain our results by random time changes and some comparison arguments with Bessel processes.

Keywords. Invariant measures, ergodicity, Bessel process, random time change.

1. Introduction

In our previous paper [6] we have studied the ergodic properties of the following class of stochastic reaction-diffusion equations

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\xi) = \mathcal{A}u(t,\xi) + f(\xi, u(t,\xi)) + g(\xi, u(t,\xi))\,\dfrac{\partial^2 w}{\partial t \partial \xi}(t,\xi), \quad t \geq 0,\ \xi \in [0,L], \\ u(0,\xi) = x(\xi), \quad \xi \in [0,L], \qquad \mathcal{B}u(t,0) = \mathcal{B}u(t,L) = 0, \quad t \geq 0, \end{cases} \tag{1.1}$$

where $w(t,\xi)$ is a Brownian sheet on $[0,\infty) \times [0,L]$, defined on some underlying complete stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, $\mathcal{A}$ is a second-order uniformly-elliptic operator given in divergence form, that is

$$\mathcal{A}h = (ah')', \qquad h \in C^2[0,L],$$

for some $a \in C^1[0,L]$ such that $a(\xi) \geq \epsilon > 0$, for any $\xi \in [0,L]$, and $\mathcal{B}$ is an operator acting on the boundary, either of Dirichlet or of general Neumann type, that is

$$\mathcal{B}h = \alpha\,(h' - \beta) + (1-\alpha)h,$$

for some $\beta \in \mathbb{R}$ and $\alpha = 0, 1$ (but with the same method it is even easier to treat the case of periodic boundary conditions).

In [6] we have assumed that the reaction term $f : [0, L] \times \mathbb{R} \to \mathbb{R}$ is continuous and locally-Lipschitz continuous with polynomial growth and some dissipation property with respect to the second variable. For the diffusion coefficient $g : [0, L] \times \mathbb{R} \to \mathbb{R}$ we have assumed that it is continuous and Lipschitz continuous with respect to the second variable. Moreover, we have assumed that for any $R > 0$

$$\inf_{\xi \in [0,L]} |g(\xi, \sigma) - g(\xi, \rho)| \geq \nu_R\, |\sigma - \rho|, \qquad \sigma, \rho \in [-R, R], \tag{1.2}$$

for some positive constant ν_R. This means that we have taken $g(\xi, \cdot)$ strictly monotone and in the case $g(\xi, \cdot)$ is differentiable we have assumed

$$\inf_{\xi \in [0,L]} \left| \frac{\partial g}{\partial \sigma}(\xi, \sigma) \right| \geq \nu := \inf_{R>0} \nu_R, \qquad \sigma \in \mathbb{R}.$$

In particular, if $\nu = 0$ we have allowed $\partial g(\xi, \sigma)/\partial \xi$ to go to zero, as $|\sigma|$ goes to infinity. But it is important to stress that, as ν_R is strictly positive for any $R > 0$, $g(\xi, \cdot)$ had no critical points, that is

$$\frac{\partial g}{\partial \sigma}(\xi, \sigma) \neq 0, \qquad (\xi, \sigma) \in [0, L] \times \mathbb{R}.$$

Under these assumptions on the coefficients f and g and under other suitable conditions, by developing a method introduced by Mueller in [15] we have proved that if u^x and u^y denote the solutions of equation (1.1), starting respectively from x and y, then

$$\lim_{t \to \infty} |u^x(t) - u^y(t)|_{L^1(0,L)} = 0, \qquad \mathbb{P} - \text{a.s.} \tag{1.3}$$

In particular we have seen how this implies that if there exists an invariant measure for (1.1), then such an invariant measure is unique, ergodic and strongly mixing.

Our aim in the present paper is to prove that the limit (1.3) holds also in the more general case of a diffusion coefficient g which is still Lipschitz continuous and strictly monotone with respect to the second variable, but can have several critical points. Namely we are going to replace condition (1.2) with the more general assumption that for any $R > 0$ there exists $\mu_R > 0$ such that

$$\inf_{\xi \in [0,L]} |g(\xi, \sigma) - g(\xi, \rho)| \geq \mu_R\, |\sigma - \rho|^\gamma, \qquad \sigma, \rho \in [-R, R], \tag{1.4}$$

for some constant $\gamma \geq 1$ independent of $R > 0$ (for more details see Hypothesis 2 and Remark 2.1 below).

It is important to notice that, as g is possibly vanishing, it is not possible to prove any smoothing effect of the transition semigroup associated with equation (1.1), so that it is not possible to apply the Doob theorem in order to prove the uniqueness of the invariant measure (for all proofs and details see [7, Chapter 4]). To this purpose, notice that the case of exponential mixing properties of some PDEs perturbed by a degenerate noise of additive type has been studied also by using the coupling method (see [8], [13] and [11] and more recently [9]).

Moreover, as we are not assuming f to be strictly dissipative, the stochastic term is not just a perturbation of a deterministic equation which *already* shows a stable behavior, unlike what happens in the important paper by Sowers [19]. This means that, without any non-degeneracy assumption on the noise, here as in [6] the stochastic part is playing a crucial role in stabilizing an equation which was not necessarily asymptotically stable, trying to extend in some sense to an infinite-dimensional setting some results on stabilization by noise proved in finite dimension (see, e.g., [1], [2], [16] and [17]; see also [3], [4], [10] and [12] for some results in infinite dimension).

It is important to stress that in the present paper we have to *pay* for the more general assumption (1.4) on the diffusion g with a more restrictive assumption on the reaction term f. Actually, while in [6] we have assumed

$$\sup_{\xi \in [0,L]} f(\xi,\sigma) - f(\xi,\rho) \leq \lambda\,(\sigma-\rho), \qquad \sigma,\rho \in \mathbb{R}, \tag{1.5}$$

for some $\lambda \in \mathbb{R}$ such that

$$\lambda < \frac{\nu_R^2}{2L}, \qquad R > 0, \tag{1.6}$$

here we assume that $f(\xi,\cdot)$ is non-increasing, for any $\xi \in [0,L]$, so that $\lambda = 0$ and in particular the inequality above is always satisfied. In any case, notice that, at least in the case of Neumann boundary conditions, the fact that $\lambda = 0$ does not mean that the deterministic part of equation (1.1) is asymptotically stable.

As we have said above, all results proved both in [6] and in the present article follow from the limiting result (1.3). In [6] we have proved (1.3) by showing that if we fix $x, y \in C[0,L]$, with $x \geq y$, and if we set

$$X(t) := Z(T(t)), \qquad t \geq 0,$$

where

$$Z(t) := \int_0^L \left[u^x(t,\xi) - u^y(t,\xi)\right]\,d\xi, \qquad t \geq 0,$$

and $T(t)$ is a suitable random time change such that $T(t) \to \infty$, $\mathbb{P}$-a.s. as $t \to \infty$, then the process $X(t)$ satisfies

$$X(t) \leq \int_0^L [x(\xi) - y(\xi)]\,d\xi + \lambda \int_0^t \frac{X(s)}{U(T(s))}\,ds + \int_0^t X(s)\,dB(s), \qquad t \geq 0, \tag{1.7}$$

where $B(t)$ is a standard Brownian motion, λ is the constant in (1.5) and $U(t)$ is a process related to $T(t)$ by

$$V(t) := \int_0^t U(s)\,ds, \qquad T(t) := V^{-1}(t), \qquad t \geq 0.$$

Actually, as $U(t)$ is strictly positive $\mathbb{P}$-a.s. (this follows mainly from condition (1.6)), by comparing $X(t)$ with a geometric Brownian motion we have shown that

$$\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} X(t) = 0, \qquad \mathbb{P}-\text{a.s.}$$

and hence we have got (1.3).

In the present paper, due to condition (1.4), instead of (1.7) we obtain

$$X(t) \leq \langle x-y \rangle + \int_0^t X(s)^\gamma \, dB(s), \qquad t \geq 0.$$

Thus, in the case $\gamma > 1$ we cannot use comparison with the geometric Brownian motion, but with the process $I(t)$, $t \geq 0$, obtained from a Bessel process $Y(t)$ of parameter $\delta := 1 + \gamma/(\gamma - 1)$ by the formula

$$Y(t) = \frac{1}{\gamma - 1} I^{1-\gamma}(t), \qquad t \geq 0.$$

This forces us to use results on the asymptotic behavior and on the set of polar points of Bessel processes.

2. Preliminaries and hypotheses

Throughout the paper we shall assume that the reaction term f satisfies the following conditions.

Hypothesis 1. The function $f : [0, L] \times \mathbb{R} \to \mathbb{R}$ is continuous. Moreover,

1. there exist $m \geq 1$ and $c \geq 0$ such that
$$\sup_{\xi \in [0,L]} |f(\xi, \sigma)| \leq c\,(1 + |\sigma|^m), \qquad \sigma \in \mathbb{R}; \tag{2.1}$$
2. if $m > 1$, there exist $a > 0$ and $c \geq 0$ such that
$$\sup_{\xi \in [0,L]} (f(\xi, \sigma + \rho) - f(\xi, \sigma))\rho \leq -a\,|\rho|^{m+1} + c\left(1 + |\sigma|^{m+1}\right), \qquad \sigma, \rho \in \mathbb{R};$$
3. for any fixed $\xi \in [0, L]$, the mapping $\sigma \mapsto f(\xi, \sigma)$ is non-increasing.

To have an idea of a class of functions f fulfilling the conditions above, assume that $h : [0, L] \times \mathbb{R} \to \mathbb{R}$ is continuous and $h(\xi, \cdot) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous with linear growth, uniformly with respect to $\xi \in [0, L]$ and assume that

$$k(\xi, \sigma) := -c(\xi)\sigma^{2n} + \sum_{j=1}^{2n} c_j(\xi)\sigma^j,$$

for some continuous coefficients $c, c_j : [0, L] \to \mathbb{R}$ such that $c(\xi) \geq c_0 > 0$, for any $\xi \in [0, L]$. Then, it is immediate to check that the mapping

$$f(\xi, \sigma) := h(\xi, \sigma) + k(\xi, \sigma), \qquad (\xi, \sigma) \in [0, L] \times \mathbb{R},$$

satisfies conditions 1. and 2. of Hypothesis 1.

As far as the diffusion coefficient g is concerned we assume the following conditions.

Hypothesis 2. The function $g : [0, L] \times \mathbb{R} \to \mathbb{R}$ is continuous. Moreover,

1. there exists $L > 0$ such that
$$\sup_{\xi \in [0,L]} |g(\xi, \sigma) - g(\xi, \rho)| \leq L\, |\sigma - \rho|, \qquad \sigma, \rho \in \mathbb{R}; \tag{2.2}$$
2. for any $R > 0$ there exists $\mu_R > 0$ such that
$$\inf_{\xi \in [0,L]} |g(\xi, \sigma) - g(\xi, \rho)| \geq \mu_R\, |\sigma - \rho|^\gamma, \qquad \sigma, \rho \in [-R, R], \tag{2.3}$$
for some $\gamma > 1$ independent of $R > 0$.

Remark 2.1. (1) The assumptions above imply that the mapping $g(\xi, \cdot)$ is Lipschitz continuous, uniformly with respect to $\xi \in [0, L]$, and it is possibly vanishing, even if there exists at most one zero, as $g(\xi, \cdot)$ is either strictly increasing or decreasing.
(2) It is important to stress that (2.3) does not imply any particular growth condition of g at infinity. In particular we can also treat the case of diffusion coefficients g which are bounded.
(3) Condition (2.3) is more general than condition (1.2) given in our previous paper [6]. Actually, assume that (1.2) holds. Then, for any $\xi \in [0, L]$ and $R > 0$, if $|t - s| \leq 1$ we have
$$|g(\xi, t) - g(\xi, s)| \geq \nu_R\, |t - s| \geq \nu_R\, |t - s|^\gamma, \qquad t, s \in [-R, R],$$
and if $|t - s| > 1$ we have
$$\begin{aligned} |g(\xi, t) - g(\xi, s)| \geq \nu_R\, |t - s| &= \nu_R\, \frac{|t - s|^\gamma}{|t - s|^{\gamma - 1} \vee 1} \\ &\geq \frac{\nu_R}{(2R)^{\gamma - 1} \vee 1}\, |t - s|^\gamma, \qquad t, s \in [-R, R]. \end{aligned}$$
This means that (2.3) holds with
$$\mu_R = \frac{\nu_R}{(2R)^{\gamma - 1} \vee 1}.$$
Moreover, as $g(\xi, \cdot)$ is Lipschitz continuous, we have that
$$\lim_{R \to \infty} \mu_R = 0.$$
(4) Condition (2.3) in Hypothesis 2 is strictly more general than condition (1.2) given in our previous paper [6]. An example of a function which does satisfy Hypothesis 2 and does not satisfy condition (1.2) is given by
$$g(\sigma) := \begin{cases} \sigma^3 & \text{if } |\sigma| \leq 1, \\ \sigma & \text{if } |\sigma| > 1. \end{cases}$$
The function g is clearly Lipschitz continuous and strictly increasing and has a zero at $\sigma = 0$. But $g'(0) = 0$, so that it is not possible to find any $\nu_1 > 0$ such that
$$|g(\sigma) - g(\rho)| \geq \nu_1\, |\sigma - \rho|, \qquad \sigma, \rho \in [-1, 1].$$

On the other side it is possible to check that there exists some $\mu > 0$ such that

$$|g(\sigma) - g(\rho)| \geq \mu \, \frac{|\sigma - \rho|^3}{|\sigma - \rho|^2 \vee 1}, \qquad \sigma, \rho \in \mathbb{R},$$

so that (2.3) is fulfilled with $\gamma = 3$.

In what follows we shall denote by A the realization in $C\,[0, L]$ of the second order operator $\mathcal{A}$ endowed with the boundary condition $\mathcal{B}$ and by e^{tA}, $t \geq 0$, the semigroup generated by A. Moreover, we shall denote by F and G the composition/multiplication operators associated with f and g respectively, that is

$$F(x)(\xi) := f(\xi, x(\xi)), \qquad [G(x)y](\xi) := g(x(\xi))y(\xi), \quad \xi \in [0, L],$$

for any $x, y : [0, L] \to \mathbb{R}$. Finally we shall set

$$w(t) = \sum_{k=1}^{\infty} e_k \beta_k(t) = \sum_{k=1}^{\infty} e_k \int_0^t \int_0^L e_k(\xi) dw(t, \xi), \qquad t \geq 0,$$

where $\{e_k\}$ is any complete orthonormal basis in $L^2(0, L)$ and $\{\beta_k\}$ is a sequence of mutually independent standard Brownian motions defined on the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

With these notations equation (1.1) can be written as the following abstract stochastic evolution equation

$$du(t) = [Au(t) + F(u(t))] \; dt + G(u(t)) dw(t), \qquad u(0) = x. \tag{2.4}$$

In [5] it has been proved that under Hypotheses 1 and 2-1. for any $x \in C\,[0, L]$ equation (1.1) admits a unique mild solution in $L^p(\Omega; C([0, T]; C\,[0, L]))$, for any $p \geq 1$ and $T > 0$. That is there exists a unique adapted process $u^x(t)$ having continuous trajectories with values in $C\,[0, L]$ such that

$$u^x(t) = e^{tA} x + \int_0^t e^{(t-s)A} F(u^x(s)) \, ds + \int_0^t e^{(t-s)A} G(u^x(s)) \, dw(s).$$

Notice that as proved in [6, Theorem 3.1] a comparison result holds for the solutions of equation (2.4). Namely, for any $x, y \in C\,[0, L]$ such that $x \geq y$ it holds

$$\mathbb{P}\left(u^x(t) \geq u^y(t), \; t \geq 0\right) = 1. \tag{2.5}$$

Moreover, we have shown that in the case $m > 1$ for any $p \geq 1$

$$\mathbb{E} \sup_{t \geq 0} |u^x(t)|^p_{C[0,L]} \leq c_p \left(1 + |x|^p_{C[0,L]}\right) \tag{2.6}$$

and

$$\sup_{t \geq t_0} \mathbb{E}\, |u^x(t)|_{C^{\theta_\star}[0,L]} < \infty,$$

for some $\theta_\star > 0$ and for any $t_0 > 0$. In particular, the family of probability measures $\{\mathcal{L}(u^x(t))\}_{t \geq t_0}$ is tight in $(C[0, L], \mathcal{B}(C\,[0, L]))$ and due to the Krylov-Bogoliubov theorem the semigroup P_t associated with equation (1.1) and defined by

$$P_t \varphi(x) = \mathbb{E}\, \varphi(u^x(t)), \qquad t \geq 0, \quad x \in C\,[0, L],$$

for any function $\varphi \in B_b(C[0,L])$, the Banach space of Borel bounded functions from $C[0,L]$ with values in $\mathbb{R}$, admits an invariant measure μ. As it is well known this means that

$$\int_{C[0,L]} P_t\varphi(x)\,d\mu(x) = \int_{C[0,L]} \varphi(x)\,d\mu(x),$$

for any $\varphi \in B_b(C[0,L])$ and $t \geq 0$.

The main result of this paper is that, under Hypotheses 1 and 2, for any $x, y \in C[0,L]$ it holds

$$\lim_{t\to\infty} |u^x(t) - u^y(t)|_{L^1(0,L)} = 0, \qquad \mathbb{P} - \text{a.s.} \tag{2.7}$$

This implies that for any $\varphi \in C_b(C[0,L])$, the Banach space of uniformly continuous and bounded functions from $C[0,L]$ with values in $\mathbb{R}$, and for any $x, y \in C[0,L]$

$$\lim_{t\to\infty} P_t\varphi(x) - P_t\varphi(y) = 0, \tag{2.8}$$

which on its turn by standard arguments implies that there exists at most one invariant measure and it is ergodic and strongly mixing.

Actually, if $\varphi \in C_b(L^1(0,L))$ then (2.7) immediately implies (2.8). If φ belongs to the larger space $C_b(C[0,L])$, as proved in [6, Lemma 5.1] there exists a sequence $\{\varphi_n\} \subset C_b(L^1(0,L))$ such that

$$\sup_{n\in\mathbb{N}} \|\varphi_n\|_{C_b(L^1(0,L))} \leq \|\varphi\|_{C_b(C[0,L])}, \qquad \lim_{n\to\infty} \varphi_n(x) = \varphi(x), \quad x \in L^1(0,L).$$

Thanks to the dominated convergence theorem this implies that (2.8) is valid also for $\varphi \in C_b(C[0,L])$.

3. The main convergence result

In the present section we prove the main result of the paper.

Theorem 3.1. *Assume that for any* $x \in C[0,L]$

$$\mathbb{P}\left(\sup_{t\geq 0} |u^x(t)|_{C[0,L]} < \infty\right) = 1. \tag{3.1}$$

Then under Hypotheses 1 *and* 2 *for any* $x, y \in C[0,L]$ *we have*

$$\lim_{t\to+\infty} |u^x(t) - u^y(t)|_{L^1(0,L)} = 0, \qquad \mathbb{P} - a.s.$$

where u^x *and* u^y *denote the solutions of* (1.1), *starting respectively from* x *and* y.

In particular, if equation (1.1) *admits an invariant measure* (*supported on* $C[0,L]$) *it is unique, ergodic and strongly mixing.*

Remark 3.2.

1. As recalled above, in [5] we have proved that if the constant m in Hypothesis 1 is strictly greater than 1, then estimate (2.6) holds, so that condition (3.1) is always satisfied.

2. In [6] we have assumed that there exists some $\lambda \in \mathbb{R}$ such that
$$\sup_{\xi \in [0,L]} f(\xi,\sigma) - f(\xi,\rho) \leq \lambda\,(\sigma-\rho), \qquad \sigma,\rho \in \mathbb{R},$$
and in the case $m > 1$ we have taken
$$\lambda < \frac{\mu_R^2}{2L}, \qquad R > 0. \tag{3.2}$$
In the present paper, as $f(\xi,\cdot)$ is non-increasing, we have that $\lambda = 0$ so that condition (3.2) is always satisfied.
3. We give a proof of the theorem above in the case of Neumann boundary conditions, as the case of Dirichlet boundary conditions can be obtained from the Neumann case as in [6, Subsection 4.2].

Let $x, y \in C\,[0,L]$ and let u^x and u^y be the unique mild solutions of equation (1.1) starting from the initial data x and y. If we set $\rho(t) := u^x(t) - u^y(t)$ and if we fix $\varphi \in C^{1,2}\,([0,\infty)\times[0,L])$ such that
$$\frac{\partial\varphi}{\partial\xi}(t,0) = \frac{\partial\varphi}{\partial\xi}(t,L) = 0,$$
as $\rho(t)$ fulfills homogeneous Neumann boundary conditions we have
$$\begin{aligned}\int_0^L \left[\rho(t,\xi)\varphi(t,\xi) - \rho(0,\xi)\varphi(0,\xi)\right] d\xi &= \int_0^t\int_0^L \rho(s,\xi)\left(\frac{\partial\varphi}{\partial t} + \mathcal{A}\varphi\right)(s,\xi)\,d\xi\,ds\\ &+ \int_0^t\int_0^L \left[f(\xi,u^x(s,\xi)) - f(\xi,u^y(s,\xi))\right]\varphi(s,\xi)\,d\xi\,ds\\ &+ \int_0^t\int_0^L \left[g(\xi,u^x(s,\xi)) - g(\xi,u^y(s,\xi))\right]\varphi(s,\xi)\,w(ds,d\xi),\end{aligned}$$
(for a proof see [20, Chapter 3]). Hence, if we take $\varphi \equiv 1$ we have
$$\begin{aligned}\int_0^L \rho(t,\xi)\,d\xi &= \int_0^L (x-y)(\xi)\,d\xi + \int_0^t\int_0^L \left[f(\xi,u^x(s,\xi)) - f(\xi,u^y(s,\xi))\right]\,d\xi\,ds\\ &+ \int_0^t\int_0^L \left[g(\xi,u^x(s,\xi)) - g(\xi,u^y(s,\xi))\right]\,w(ds,d\xi).\end{aligned}$$
Thus, by setting for any $t \geq 0$
$$Z(t) := \int_0^L \rho(t,\xi)\,d\xi, \qquad C(t) := \int_0^L \left[f(\xi,u^x(t,\xi)) - f(\xi,u^y(t,\xi))\right]\,d\xi$$
and
$$M(t) := \int_0^t\int_0^L \left[g(\xi,u^x(s,\xi)) - g(\xi,u^y(s,\xi))\right]\,w(ds,d\xi),$$
we have
$$Z(t) = \langle x-y\rangle + \int_0^t C(s)\,ds + M(t), \tag{3.3}$$
where $\langle x-y\rangle$ denotes the mean value of $x-y$ in the interval $[0,L]$.

Next, we analyze the term $M(t)$. It is clear that $\{M(t)\}_{t\geq 0}$ is a $\mathcal{G}_t$-martingale, where $\{\mathcal{G}_t\}_{t\geq 0}$ is the filtration defined by

$$\mathcal{G}_t := \sigma\left(\int_0^\infty\int_0^L v(s,\xi)w(ds,d\xi),\ v\in\ L^2([0,\infty)\times[0,L]),\,\operatorname{supp} v\subseteq[0,t]\times[0,L]\right),$$

for any $t\geq 0$. In the next lemma we describe the quadratic variation of $M(t)$.

Lemma 3.3. *Under Hypotheses* 1 *and* 2 *there exists an adapted process* $U(t)$ *such that*

$$\frac{d\,\langle M\rangle_t}{dt} = Z^{2\gamma}(t)U(t), \qquad t\geq 0, \tag{3.4}$$

and

$$\mathbb{P}\left(L^{1-2\gamma}\,\Lambda\leq U(t)<+\infty,\ t\geq 0\right)=1, \tag{3.5}$$

where Λ *is the random variable defined by*

$$\Lambda := \sum_{n=1}^\infty \mu_n^2\, I_{A_n},$$

and A_n *is defined for each* $n\in\mathbb{N}$ *by*

$$A_n := \left\{\sup_{t\geq 0}|u^x(t)|_{C[0,L]}\vee\sup_{t\geq 0}|u^y(t)|_{C[0,L]}\in\ [n-1,n)\right\}. \tag{3.6}$$

Proof. We have

$$\langle M\rangle_t = \int_0^t\int_0^L\left[g(\xi,u^x(s,\xi))-g(\xi,u^y(s,\xi))\right]^2\,d\xi\,ds$$

and hence

$$\frac{d\,\langle M\rangle_t}{dt} = \int_0^L\left[g(\xi,u^x(t,\xi))-g(\xi,u^y(t,\xi))\right]^2\,d\xi.$$

According to condition (2.3) in Hypothesis 2, if

$$\sup_{t\geq 0}|u^x(t)|_{C[0,L]}\vee\sup_{t\geq 0}|u^y(t)|_{C[0,L]}\in\ [n-1,n),$$

this yields

$$\frac{d\,\langle M\rangle_t}{dt}\geq\mu_n^2\int_0^L\rho(t,\xi)^{2\gamma}\,d\xi\geq L^{1-2\gamma}\mu_n^2\left(\int_0^L\rho(t,\xi)\,d\xi\right)^{2\gamma} = L^{1-2\gamma}\,Z^{2\gamma}(t),$$

so that, if A_n is the event defined in (3.6)

$$\frac{d\,\langle M\rangle_t}{dt}\geq L^{1-2\gamma}\sum_{n=1}^\infty\mu_n^2\,I_{A_n}\,Z^{2\gamma}(t) = L^{1-2\gamma}\Lambda\,Z^{2\gamma}(t)\qquad t\geq 0.$$

Moreover, according to condition (2.2) in Hypothesis 2 we have

$$\frac{d\,\langle M\rangle_t}{dt}\leq L^2\int_0^L\rho^2(t,\xi)\,d\xi = L^2\,|\rho(t)|^2_{L^2(0,L)}. \tag{3.7}$$

Since $\rho(t,\xi) \geq 0$ for any $t \geq 0$ and $\xi \in [0,L]$, $\mathbb{P}$-a.s. (see the comparison result (2.5)), this means

$$Z(t) = \int_0^L \rho(t,\xi)\,d\xi = |\rho(t)|_{L^1(0,L)} = 0 \Longrightarrow \frac{d\,\langle M\rangle_t}{dt} = 0.$$

Then, if we set

$$U(t) := \begin{cases} Z(t)^{-2\gamma}\,\dfrac{d\,\langle M\rangle_t}{dt} & \text{if } Z(t) \neq 0,\\ L^{1-2\gamma}\,\Lambda & \text{if } Z(t) = 0,\end{cases}$$

we have that $U(t)$ is an adapted process which fulfills (3.4) and such that

$$\mathbb{P}\left(U(t) \geq L^{1-2\gamma}\,\Lambda,\ t \geq 0\right) = 1.$$

Now, we conclude the proof of the lemma by noticing that thanks to (3.4) and (3.7)

$$\begin{aligned}\{\exists\, t \geq 0\,:\, U(t) = +\infty\} &= \left\{\exists\, t \geq 0\,:\, \frac{d\,\langle M\rangle_t}{dt} = +\infty\right\}\\ &\subseteq \left\{\exists\, t \geq 0\,:\, |\rho(t)|_{L^2(0,L)} = +\infty\right\},\end{aligned}$$

and then, since $\rho \in L^p(\Omega; C([0,T]; C[0,L]))$, for any $T > 0$, we have

$$\mathbb{P}\left(U(t) < +\infty,\ t \geq 0\right) = 1.$$

□

Due to (3.1) and (3.5) we have that

$$\mathbb{P}\left(U(t) > 0,\ t \geq 0\right) \geq \mathbb{P}\left(\sup_{t\geq 0}|u^x(t)|_{C[0,L]} \vee \sup_{t \geq 0}|u^y(t)|_{C[0,L]} < \infty\right) = 1. \tag{3.8}$$

Hence, if we define

$$\varphi(t) := \int_0^t U(s)\,ds, \qquad t \geq 0,$$

we have that $\varphi(t)$ is a $\mathbb{P}$-a.s. strictly increasing adapted process with continuous trajectories, such that $\varphi(0) = 0$. Moreover, if

$$\omega \in \bigcup_{n=1}^\infty A_n = \{\,\sup_{t\geq 0}|u^x(t)|_{C[0,L]} \vee \sup_{t \geq 0}|u^y(t)|_{C[0,L]} < \infty\,\},$$

we have that there exists $\bar{n} \in \mathbb{N}$ such that $U(t)(\omega) \geq L^{1-2\gamma}\,\mu_{\bar{n}}^2$, for any $t \geq 0$, so that

$$\lim_{t\to\infty}\varphi(t)(\omega) = \lim_{t\to\infty}\int_0^t U(s)(\omega)\,ds \geq \lim_{t\to\infty} L^{1-2\gamma}\,\mu_{\bar{n}}^2\,t = \infty.$$

This means that

$$\lim_{t\to\infty}\varphi(t) = \infty, \qquad \mathbb{P}-\text{a.s.},$$

and hence it is possible to define the inverse of φ

$$T(t) := \varphi^{-1}(t), \quad t \geq 0.$$

In this way $T(t)$ defines a good random time change, so that the process $X(t) := Z(T(t))$ is well defined for any $t \geq 0$ and

$$\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} X(t), \qquad \mathbb{P}-\text{a.s.} \tag{3.9}$$

In particular we have that $Z(t)$ converges $\mathbb{P}$-a.s. to zero as t goes to ∞, if and only if the process $X(t) = Z(T(t))$ converges $\mathbb{P}$-a.s. to zero as t goes to ∞.

Lemma 3.4. *Fix $x, y \in C\,[0,L]$, with $x \geq y$. Then, under the same conditions of Theorem* 3.1 *we have that there exists a Brownian motion $B(t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$X(t) = \langle x - y\rangle + \int_0^t \frac{C(T(s))}{U(T(s))}\,ds + \int_0^t X(s)^\gamma\,dB(s).$$

Proof. We substitute t with $T(t)$ in (3.3) and obtain

$$X(t) = Z(T(t)) = \langle x - y\rangle + \int_0^{T(t)} C(s)\,ds + M(T(t)), \qquad t \geq 0.$$

If we set $r = T^{-1}(s) = \varphi(s)$, we have $dr = d\varphi(s) = U(s)ds = U(T(r))ds$ and hence

$$\int_0^{T(t)} C(s)\,ds = \int_0^t \frac{C(T(r))}{U(T(r))}\,dr, \qquad t \geq 0.$$

Concerning the martingale term, from (3.4) with an analogous change of variable we have

$$\langle M\rangle_{T(t)} = \int_0^{T(t)} Z^{2\gamma}(s)U(s)\,ds = \int_0^t Z^{2\gamma}(T(s))\,ds, \qquad t \geq 0.$$

Now, as we have recalled in [6], the process $N(t) := M(T(t)), t \geq 0$, is a $\left\{\mathcal{G}_{T(t)}\right\}_{t\geq 0}$ martingale (for a proof see, e.g., [18, Proposition V.1.5]) such that

$$d\,\langle N\rangle_t = d\,\langle M\rangle_{T(t)} = Z^{2\gamma}(T(t)), \qquad t \geq 0.$$

Therefore, as proved, e.g., in [18, Proposition V.3.8 and following remarks], it is possible to fix a standard Brownian motion $B(t)$ such that

$$M(T(t)) = N(t) = \int_0^t Z^\gamma(T(s))\,dB(s) = \int_0^t X^\gamma(s)\,dB(s), \qquad t \geq 0.$$

Collecting all terms we conclude the proof of the lemma. □

Now, for any $\gamma > 1$ we denote by $I(t)$ the solution of the stochastic equation

$$dI(t) = I^\gamma(t)\,dB(t), \qquad I(0) > 0,$$

where $B(t)$ is some standard Brownian motion. Notice that if one defines

$$Y(t) := \frac{1}{\gamma - 1}\,I^{1-\gamma}(t), \qquad t \geq 0,$$

by Itô's formula $Y(t)$ solves the problem

$$dY(t) = \frac{\gamma}{2(\gamma-1)}\,\frac{1}{Y(t)}\,dt - dB(t), \qquad Y(0) = \frac{1}{\gamma-1}I^{1-\gamma}(0),$$

so that $Y(t)$ is a Bessel process of dimension $\delta := 1+\gamma/(\gamma-1) > 2$ (for all details see [18, Chapter 11]). In particular, if $\delta \in \mathbb{N}$

$$Y(t) = |W(t)|_{\mathbb{R}^\delta}, \qquad t \geq 0,$$

for some δ-dimensional Brownian motion $W(t)$. Note that for any $\delta \geq 2$ the set $\{0\}$ is polar for the Bessel process of dimension δ and then

$$\mathbb{P}\left(\sup_{t\geq 0} I(t) < \infty\right) = \mathbb{P}\left(\inf_{t\geq 0} Y(t) > 0\right) = 1. \tag{3.10}$$

As shown in [14, Theorem 3.1], with this scale change we have

$$\limsup_{t\to\infty} \frac{t^{\frac{1}{2(\gamma-1)}}}{(\log t)^{1+\epsilon}} I(t) = 0, \qquad \mathbb{P}-\text{a.s.}$$

for any $\epsilon > 0$. As

$$\lim_{t\to\infty} \frac{t^{\frac{1}{2(\gamma-1)}}}{(\log t)^{1+\epsilon}} = \infty,$$

this implies that

$$\limsup_{t\to\infty} I(t) = 0, \qquad \mathbb{P}-\text{a.s.} \tag{3.11}$$

In the next lemma we compare the processes $X(t) = Z(T(t))$ and $I(t)$, in order to describe the asymptotic behavior of $X(t)$ by means of the asymptotic behavior of $I(t)$ which we have just described. For this purpose, if we set

$$\widetilde{C}(t) := \frac{C(T(t))}{U(T(t))}, \qquad t \geq 0, \tag{3.12}$$

due to (3.1) and (3.8) we have that under the same conditions of Lemma 3.4

$$\mathbb{P}\left(\sup_{t\geq 0} |\widetilde{C}(t)| < \infty\right) = 1.$$

Moreover, according to the comparison estimate (2.5), as $x \geq y$ and $f(\xi,\cdot)$ is non-increasing for any fixed $\xi \in [0,L]$,

$$\mathbb{P}\left(\int_0^L \left[f(\xi,u^x(t)) - f(\xi,u^y(t))\right] d\xi \leq 0, \ \ t \geq 0\right) = 1.$$

Hence, recalling how we have defined $C(t)$, we get

$$\mathbb{P}\left(\widetilde{C}(t) \leq 0, \ t \geq 0\right) = 1. \tag{3.13}$$

Lemma 3.5. *Let us fix $x, y \in C\,[0,L]$, with $x \geq y$. Then*

$$\mathbb{P}\,(\,X(t) \leq I(t), \ t \geq 0\,) = 1.$$

Proof. Assume by contradiction that there exists some $T > 0$ such that

$$\mathbb{P}\left(\sup_{t\in[0,T]} X(t) - I(t) > 0\right) > 0.$$

As $X(t) = Z(T(t)) = |\rho(T(t))|_{L^1(0,L)}$, due to (3.1) we have

$$\mathbb{P}\left(\sup_{t\geq 0} X(t) < \infty\right) = 1.$$

Then, thanks to (3.10) we have

$$\lim_{R\to\infty} \mathbb{P}\left(\sup_{t\geq 0} X(t) \vee \sup_{t\geq 0} I(t) \leq R\right) = 1,$$

so that we can fix $\bar{R} < 0$ such that

$$\mathbb{P}\left(\sup_{t\in[0,T]} X(t) - I(t) > 0, \ \ \sup_{t\geq 0}|X(t)| \vee \sup_{t\geq 0}|I(t)| \leq \bar{R}\right) > 0. \tag{3.14}$$

Now, as the function $\sigma(x) = x^\gamma$ is locally Lipschitz continuous, there exists a sequence of Lipschitz continuous mappings $\{\sigma_R\}_{R>0}$ such that $\sigma_R \equiv \sigma$ on $[-R, R]$. If for any $R > 0$ we denote by X_R the solution of the problem

$$dX_R(t) = \widetilde{C}(t)\,dt + \sigma_R(X_R(t))\,dB(t), \qquad X_R(0) = \langle x - y\rangle\,,$$

and denote by I_R the solution of the problem

$$dI_R(t) = \sigma_R(I_R(t))\,dB(t), \qquad I_R(0) = \langle x - y\rangle\,,$$

since the coefficients are Lipschitz and (3.13) holds, as proved for example in [18, Theorem 3.7]

$$\mathbb{P}\,(\,X_R(t) \leq I_R(t),\ t \geq 0) = 1. \tag{3.15}$$

On the other hand, since for any $R > 0$

$$\sup_{t\geq 0}|X(t)| \vee \sup_{t\geq 0}|I(t)| \leq R \Longrightarrow X_R \equiv X, \ \ I_R \equiv I,$$

due to (3.14) we have

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T]} X_{\bar{R}}(t) - I_{\bar{R}}(t) > 0\right)\\
&\quad\geq \mathbb{P}\left(\sup_{t\in[0,T]} X_{\bar{R}}(t) - I_{\bar{R}}(t) > 0, \ \ \sup_{t\geq 0}|X(t)| \vee \sup_{t\geq 0}|I(t)| \leq \bar{R}\right)\\
&\quad= \mathbb{P}\left(\sup_{t\in[0,T]} X(t) - I(t) > 0, \ \ \sup_{t\geq 0}|X(t)| \vee \sup_{t\geq 0}|I(t)| \leq \bar{R}\right) > 0,
\end{aligned}$$

and this contradicts (3.15). $\square$

As a consequence of the previous lemma and of (3.9) and (3.11) it follows that for any $x, y \in C[0, L]$, with $x \geq y$,

$$\lim_{t\to\infty} |u^x(t) - u^y(t)|_{L^1(0,L)} = \lim_{t\to\infty} Z(t) = \lim_{t\to\infty} X(\varphi(t)) = 0, \qquad \mathbb{P} - \text{a.s.}$$

As for any $x, y \in C[0, L]$

$$|u^x(t) - u^y(t)|_{L^1(0,L)} \leq |u^x(t) - u^{x\wedge y}(t)|_{L^1(0,L)} + |u^{x\wedge y}(t) - u^y(t)|_{L^1(0,L)},$$

we can conclude that Theorem 3.1 holds for any $x, y \in C[0, L]$.

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Kolmogorov Operators of Hamiltonian Systems Perturbed by Noise

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A Philippe avec nos amitiées sincères

Abstract. We consider a second-order elliptic operator $\mathcal{K}$ arising from Hamiltonian systems with friction in $\mathbb{R}^{2n}$ perturbed by noise. An invariant measure for this operator is $\mu(dx, dy) = \exp(-2H(x, y))dx\, dy$, where H is the Hamiltonian. We study the realization $K : H^2(\mathbb{R}^{2n}, \mu) \mapsto L^2(\mathbb{R}^{2n}, \mu)$ of $\mathcal{K}$ in $L^2(\mathbb{R}^{2n}, \mu)$, proving that it is m-dissipative and that it generates an analytic semigroup.

1. Introduction

We consider a Hamiltonian system perturbed by noise,

$$\begin{cases} dX(t) = D_y H(X(t), Y(t))dt + \sqrt{\alpha}\, dW_1(t), \\ dY(t) = -D_x H(X(t), Y(t))dt + \sqrt{\beta}\, dW_2(t), \end{cases} \tag{1}$$

where $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ is the Hamiltonian which we assume to be regular, nonnegative but not necessarily Lipschitz continuous, W_1, W_2 are independent n-dimensional Brownian motions and α, β are positive constants. The Kolmogorov operator corresponding to (1) is

$$\mathcal{K}\varphi = \frac{\alpha}{2}\, \Delta_x \varphi + \frac{\beta}{2}\, \Delta_y \varphi + \langle D_y H, D_x \varphi\rangle - \langle D_x H, D_y \varphi\rangle, \tag{2}$$

where $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_y$, and Δ_x, Δ_y, D_x, D_y are the Laplacian and the gradient with respect to the variables x, y only. It is easy to see that

$$\int_{\mathbb{R}^{2n}} \mathcal{K}\varphi\, dx\, dy = 0$$

for each test function φ, which means that the Lebesgue measure in $\mathbb{R}^{2n}$ is invariant for $\mathcal{K}$. It is well known that, in order that an invariant probability measure exists, some *friction* term must be added, see, e.g., [5]. In this paper we want to consider the case that the probability measure has density (with respect to the Lebesgue measure) proportional to $e^{-2H(x,y)}$. So, we assume that

$$Z := \int_{\mathbb{R}^{2n}} \exp(-2H)dx\,dy < +\infty \tag{3}$$

and we set

$$\mu(dx, dy) = Z^{-1}e^{-2H(x,y)}dxdy.$$

The simplest way to let μ be invariant is to consider the modified system

$$\begin{cases} dX(t) = D_yH(X(t), Y(t))dt - \alpha D_xH(X(t), Y(t))dt + \sqrt{\alpha}\, dW_1(t), \\ dY(t) = -D_xH(X(t), Y(t))dt - \beta D_yH(X(t), Y(t))dt + \sqrt{\beta}\, dW_2(t). \end{cases} \tag{4}$$

In this case the Kolmogorov operator is

$$\mathcal{K}\varphi = \frac{\alpha}{2}\,\Delta_x\varphi + \frac{\beta}{2}\,\Delta_y\varphi + \langle D_yH - \alpha D_xH, D_x\varphi\rangle - \langle D_xH + \beta D_yH, D_y\varphi\rangle, \tag{5}$$

and

$$\int_{\mathbb{R}^{2n}} \mathcal{K}\varphi\, e^{-2H(x,y)}dxdy = 0,$$

for each test function φ. Therefore, the measure μ is invariant for $\mathcal{K}$.

Our aim is to study the realization K of $\mathcal{K}$ in $L^2(\mathbb{R}^{2n}, \mu)$. The main result is that if the second-order derivatives of H are bounded, then

$$K : D(K) = H^2(\mathbb{R}^{2n}, \mu) \mapsto L^2(\mathbb{R}^{2n}, \mu), \quad K\varphi = \mathcal{K}\varphi$$

is an m-dissipative operator. Therefore, it generates a strongly continuous contraction semigroup in $L^2(\mathbb{R}^{2n}, \mu)$.

Note that K is not symmetric, but we can show that for all $\varphi \in H^2(\mathbb{R}^{2n}, \mu)$, $\psi \in H^1(\mathbb{R}^{2n}, \mu)$ we have

$$\begin{aligned} \int_{\mathbb{R}^{2n}} K\varphi\,\psi\, d\mu = &-\frac{1}{2}\int_{\mathbb{R}^{2n}} (\alpha\langle D_x\varphi, D_x\psi\rangle + \beta\langle D_y\varphi, D_y\psi\rangle)d\mu \\ &+\frac{1}{2}\int_{\mathbb{R}^{2n}} (\langle 2H_y - \alpha H_x, D_x\varphi\rangle + \langle -2H_x + \beta H_y, D_y\varphi\rangle)\psi\, d\mu. \end{aligned} \tag{6}$$

However, taking $\psi = \varphi$ in (6) and manipulating the last integral (or else, integrating the equality $\mathcal{K}(\varphi^2) = 2\varphi\mathcal{K}\varphi + \alpha|D_x\varphi|^2 + \beta|D_y\varphi|^2$ with respect to μ and recalling that $\int_{\mathbb{R}^{2n}} \mathcal{K}(\varphi^2)d\mu = 0$), we get

$$\int_{\mathbb{R}^{2n}} K\varphi\,\varphi\, d\mu = -\frac{1}{2}\int_{\mathbb{R}^{2n}} (\alpha|D_x\varphi|^2 + \beta|D_y\varphi|^2)d\mu \tag{7}$$

which is a crucial formula for our analysis. It implies immediately that K is dissipative.

A typical procedure for showing m-dissipativity is to define a (dissipative) realization K_0 of $\mathcal{K}$ in $L^2(\mathbb{R}^{2n}, \mu)$ with a small domain, say for instance $C^2_b(\mathbb{R}^{2n})$ (the space of the continuous bounded functions with continuous and bounded first- and second-order derivatives), and to prove that the range of $\lambda\mathbb{I} - K_0$ is dense in

$L^2(\mathbb{R}^{2n},\mu)$ for some $\lambda > 0$. Then by the Lumer-Phillips theorem the closure $\overline{K}_0$ of K_0 is an m-dissipative operator. But the problem of the characterization of the domain of $\overline{K}_0$ still remains open. So, we take the stick from the other side. We define the above realization K of $\mathcal{K}$ on $H^2(\mathbb{R}^{2n},\mu)$, and we show that K is m-dissipative. Since the H^2 norm is equivalent to the graph norm of K, from general density results in weighted Sobolev spaces it follows that $C_b^2(\mathbb{R}^{2n})$ and $C_0^\infty(\mathbb{R}^{2n})$ are cores for K.

From the point of view of the theory of elliptic PDE's, we show that for each $\lambda > 0$ and $f \in L^2(\mathbb{R}^{2n},\mu)$, the equation

$$\lambda u - \mathcal{K}u = f$$

has a unique solution $u \in H^2(\mathbb{R}^{2n},\mu)$. So, we have a maximal regularity result in weighted L^2 spaces. In fact, we prove something more: if a function $u \in H^2_{loc}(\mathbb{R}^{2n})$ satisfies $\lambda u - \mathcal{K}u = f \in L^2(\mathbb{R}^{2n},\mu)$, then $u \in H^2(\mathbb{R}^{2n},\mu)$ and

$$\|u\|_{L^2(\mathbb{R}^{2n},\mu)} \leq \frac{1}{\lambda}\,\|f\|_{L^2(\mathbb{R}^{2n},\mu)},$$

which is the dissipativity estimate,

$$\int_{\mathbb{R}^{2n}} (\alpha|D_x\varphi|^2 + \beta|D_y\varphi|^2)d\mu \leq \frac{2}{\lambda}\,\|f\|^2_{L^2(\mathbb{R}^{2n},\mu)},$$

which comes from (7),

$$\int_{\mathbb{R}^{2n}} |D^2\varphi|^2 d\mu \leq C\|f\|^2_{L^2(\mathbb{R}^{2n},\mu)},$$

which is not obvious. Here C depends on λ and on the sup norm of the second derivatives of H.

After the elliptic regularity result we prove also some properties of the semigroup $T(t)$ generated by K. First, we show that it is an analytic semigroup. In general, elliptic operators with Lipschitz continuous unbounded coefficients do not generate analytic semigroups in L^2 spaces with respect to invariant measures. See a counterexample in [9]. In our case we can prove that the L^2 norm of the second-order space derivatives of $T(t)\varphi$ blows up as $Ct^{-1}\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)}$, as $t \to 0$, and since the graph norm of K is equivalent to the H^2 norm, then $\|tKT(t)\varphi\|_{L^2(\mathbb{R}^{2n},\mu)}$ is bounded by *const.* $\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)}$ near $t = 0$.

Second, we address to asymptotic behavior of $T(t)$, proving that $T(t)\varphi$ weakly converges to the mean value $\overline{\varphi}$ as $t \to +\infty$ for each $\varphi \in L^2(\mathbb{R}^{2n},\mu)$ (strongly mixing property).

2. m-dissipativity of K

Throughout this section we assume that $H : \mathbb{R}^{2n} \mapsto \mathbb{R}$ is a nonnegative C^2 function with bounded second-order derivatives, such that (3) holds.

The Hilbert spaces $H^1(\mathbb{R}^{2n},\mu)$, $H^2(\mathbb{R}^{2n},\mu)$ are defined as the sets of all $u \in H^1_{loc}(\mathbb{R}^{2n})$ (respectively, $u \in H^2_{loc}(\mathbb{R}^{2n})$), such that u and its first-order derivatives (resp. u and its first- and second-order derivatives) belong to $L^2(\mathbb{R}^{2n},\mu)$. It is easy to see that $C_0^\infty(\mathbb{R}^{2n})$, the space of the smooth functions with compact support, is dense in $L^2(\mathbb{R}^{2n},\mu)$, in $H^1(\mathbb{R}^{2n},\mu)$, and in $H^2(\mathbb{R}^{2n},\mu)$. A proof is in [4, Lemma 2.1].

The main result of this paper is that

$$K\colon D(K) := H^2(\mathbb{R}^{2n},\mu) \mapsto L^2(\mathbb{R}^{2n},\mu), \quad K\varphi = \mathcal{K}\varphi$$

is m-dissipative. To this aim we need an embedding lemma.

Lemma 2.1. *For every $\varphi \in H^1(\mathbb{R}^{2n},\mu)$ and $i = 1,\dots,n$ the functions $\varphi D_{x_i}H$ and $\varphi D_{y_i}H$ belong to $L^2(\mathbb{R}^{2n},\mu)$. Moreover*

$$\int_{\mathbb{R}^{2n}} (\varphi D_{x_i}H)^2 d\mu \le \int_{\mathbb{R}^{2n}} (\|D_{x_ix_i}H\|_\infty \varphi^2 + (D_{x_i}\varphi)^2) d\mu,$$

$$\int_{\mathbb{R}^{2n}} (\varphi D_{y_i}H)^2 d\mu \le \int_{\mathbb{R}^{2n}} (\|D_{y_iy_i}H\|_\infty \varphi^2 + |D_{y_i}\varphi|^2) d\mu,$$

for all $\varphi \in H^1(\mathbb{R}^{2n},\mu)$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^{2n})$. For every $i = 1,\dots,n$ we have

$$\begin{aligned}\int_{\mathbb{R}^{2n}} (\varphi D_{x_i}H)^2 d\mu &= \frac{1}{2Z}\int_{\mathbb{R}^{2n}} \varphi^2 D_{x_i}H(-D_{x_i}e^{-2H})dx\,dy\\
&= \frac{1}{2Z}\int_{\mathbb{R}^{2n}} (\varphi^2 D_{x_ix_i}H + 2\varphi\, D_{x_i}\varphi\, D_{x_i}H)e^{-2H}dx\,dy\\
&\le \frac{\|D_{x_ix_i}H\|_\infty}{2}\|\varphi\|^2_{L^2(\mathbb{R}^{2n},\mu)} + \|\varphi D_{x_i}H\|_{L^2(\mathbb{R}^{2n},\mu)}\|D_{x_i}\varphi\|_{L^2(\mathbb{R}^{2n},\mu)}\\
&\le \frac{\|D_{x_ix_i}H\|_\infty}{2}\|\varphi\|^2_{L^2(\mathbb{R}^{2n},\mu)} + \frac{1}{2}\|\varphi D_{x_i}H\|^2_{L^2(\mathbb{R}^{2n},\mu)} + \frac{1}{2}\|D_{x_i}\varphi\|^2_{L^2(\mathbb{R}^{2n},\mu)}\end{aligned}$$

so that

$$\|\varphi D_{x_i}H\|^2_{L^2(\mathbb{R}^{2n},\mu)} \le \|D_{x_ix_i}H\|_\infty \|\varphi\|^2_{L^2(\mathbb{R}^{2n},\mu)} + \|D_{x_i}\varphi\|^2_{L^2(\mathbb{R}^{2n},\mu)}.$$

Similarly,

$$\|\varphi D_{y_i}H\|^2_{L^2(\mathbb{R}^{2n},\mu)} \le \|D_{y_iy_i}H\|_\infty \|\varphi\|^2_{L^2(\mathbb{R}^{2n},\mu)} + \|D_{y_i}\varphi\|^2_{L^2(\mathbb{R}^{2n},\mu)},$$

and since $C_0^\infty(\mathbb{R}^{2n})$ is dense in $H^1(\mathbb{R}^{2n},\mu)$ the statement follows. □

The lemma has several important consequences. It implies that for every $\varphi \in H^2(\mathbb{R}^{2n},\mu)$ the drift $\langle D_yH, D_x\varphi\rangle - \langle D_xH, D_y\varphi\rangle$ belongs to $L^2(\mathbb{R}^{2n},\mu)$. It implies also (taking $\varphi \equiv 1$) that $|DH| \in L^2(\mathbb{R}^{2n},\mu)$. Moreover, in the case that $|DH| \to +\infty$ as $|(x,y)| \to +\infty$, the estimate

$$\int_{\mathbb{R}^{2n}} \varphi^2|DH|^2 d\mu \le C\|\varphi\|_{H^1(\mathbb{R}^{2n},\mu)}, \quad \varphi \in H^1(\mathbb{R}^{2n},\mu),$$

implies easily that $H^1(\mathbb{R}^{2n},\mu)$ is compactly embedded in $L^2(\mathbb{R}^{2n},\mu)$. See, e.g., [7, Prop. 3.4].

The integration formula (7) implies immediately that K is dissipative. What is not trivial is m-dissipativity. To prove m-dissipativity we have to solve the resolvent equation

$$\lambda\varphi - \mathcal{K}\varphi = f \tag{8}$$

for each $f \in L^2(\mathbb{R}^{2n}, \mu)$ and $\lambda > 0$, and show that the solution φ belongs to $H^2(\mathbb{R}^{2n}, \mu)$. That is, we have to prove an existence and maximal regularity result for an elliptic equation with unbounded coefficients. Of course it is enough to prove that for each $f \in C_0^\infty(\mathbb{R}^{2n})$ the resolvent equation has a unique solution φ in $H^2(\mathbb{R}^{2n}, \mu)$, and $\|\varphi\|_{H^2(\mathbb{R}^{2n},\mu)} \leq C\|f\|_{L^2(\mathbb{R}^{2n},\mu)}$.

Since the coefficients of $\mathcal{K}$ are regular enough, existence of a solution may be proved in several ways. For instance, the problem in the whole $\mathbb{R}^{2n}$ may be approached by a sequence of Dirichlet problems in the balls $B(0,k)$, and using classical interior estimates for solutions of second-order elliptic problems we arrive at a solution $\varphi \in H^2_{loc}(\mathbb{R}^{2n})$. See, e.g., [8, Theorem 3.4]. Or else, we may use the stochastic characteristics method, that gives a solution to (8) as

$$\varphi = \int_0^{+\infty} e^{-\lambda t}\mathbb{E}[f(X(t), Y(t))]dt$$

where $(X(t), Y(t))$ is the solution to (4) with initial data $X(0) = x$, $Y(0) = y$. See, e.g., [3, Thms. 1.2.5, 1.6.6]. The assumptions of [3] are satisfied because the second-order derivatives of H are bounded, and consequently the first-order derivatives of H have at most linear growth.

Uniqueness of the solution in $H^2(\mathbb{R}^{2n}, \mu)$ follows immediately from dissipativity. Estimates for the second-order derivatives in $L^2(\mathbb{R}^{2n}, \mu)$ are less obvious. They are proved in the next theorem.

Theorem 2.2. *Let $\varphi \in H^2_{loc}(\mathbb{R}^{2n})$ satisfy (8), with $f \in L^2(\mathbb{R}^{2n}, \mu)$. Then*

$$\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)} \leq \frac{1}{\lambda}\,\|f\|_{L^2(\mathbb{R}^{2n},\mu)} \tag{9}$$

$$\int_{\mathbb{R}^{2n}} (\alpha|D_x\varphi|^2 + \beta|D_y\varphi|^2)d\mu \leq \frac{2}{\lambda}\,\|f\|^2_{L^2(\mathbb{R}^{2n},\mu)} \tag{10}$$

$$\int_{\mathbb{R}^{2n}} |D^2\varphi|^2 d\mu \leq C\|f\|^2_{L^2(\mathbb{R}^{2n},\mu)} \tag{11}$$

where C does not depend on f and φ.

Proof. Without loss of generality we may assume that $f \in C_0^\infty(\mathbb{R}^{2n})$. Then $\varphi \in H^3_{loc}(\mathbb{R}^{2n})$, by local elliptic regularity. Moreover by the Schauder estimates of [6], φ and its first- and second-order derivatives are bounded and Hölder continuous. In particular, $\varphi \in H^2(\mathbb{R}^{2n}, \mu)$.

Estimates (9) and (10) follow in a standard way, multiplying both sides of (8) by φ and using the integration formula (7). To get (11) we differentiate (8) with respect to x_i and y_i, obtaining

$$\begin{aligned}\lambda D_{x_i}\varphi - KD_{x_i}\varphi - \sum_{k=1}^{n}(D_{x_iy_k}H - \alpha D_{x_ix_k}H)D_{x_k}\varphi& \\ + \sum_{k=1}^{n}(D_{x_ix_k}H + \beta D_{x_iy_k}H)D_{y_k}\varphi &= D_{x_i}f, \\ \lambda D_{y_i}\varphi - KD_{y_i}\varphi - \sum_{k=1}^{n}(D_{y_iy_k}H - \alpha D_{y_ix_k}H)D_{x_k}\varphi& \\ + \sum_{k=1}^{n}(D_{y_ix_k}H + \beta D_{y_iy_k}H)D_{y_k}\varphi &= D_{y_i}f.\end{aligned}$$

Replacing the expressions of the derivatives of f in the equality

$$\begin{aligned}\int_{\mathbb{R}^{2n}} K\varphi\, f\, d\mu = &-\frac{1}{2}\int_{\mathbb{R}^{2n}}(\alpha\langle D_x\varphi, D_xf\rangle + \beta\langle D_y\varphi, D_yf\rangle)d\mu \\ &+ \int_{\mathbb{R}^{2n}}(\langle D_yH, D_x\varphi\rangle - \langle D_xH, D_y\varphi\rangle)f\, d\mu\end{aligned}$$

that follows from (6), we obtain

$$\begin{aligned}\int_{\mathbb{R}^{2n}} K\varphi\, f\, d\mu &= \int_{\mathbb{R}^{2n}}(\lambda\varphi - f)f d\mu \\ &= \int_{\mathbb{R}^{2n}} -\frac{\lambda}{2}(\alpha|D_x\varphi|^2 + \beta|D_y\varphi|^2) \\ &\quad + \frac{1}{2}\sum_{i=1}^{n}(\alpha D_{x_i}\varphi\, KD_{x_i}\varphi + \beta D_{y_i}\varphi\, KD_{y_i}\varphi)d\mu \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^{2n}} -\langle SD\varphi, D\varphi\rangle + (\langle D_yH, D_x\varphi\rangle - \langle D_xH, D_y\varphi\rangle)f\, d\mu,\end{aligned}$$

where the $2n \times 2n$ matrix S has entries

$$\begin{aligned}S_{i,k} &= -\frac{\alpha}{2}(D_{x_iy_k}H - \alpha D_{x_ix_k}H), \\ S_{i,n+k} = S_{n+i,k} &= \frac{\alpha}{2}D_{x_ix_k}H - \frac{\beta}{2}D_{y_iy_k}H + \alpha\beta D_{x_iy_k}H, \\ S_{n+i,n+k} &= \frac{\beta}{2}(D_{y_ix_k}H + \beta D_{y_iy_k}H),\end{aligned}$$

for i, $k = 1, \dots, n$.

Using formula (6) in the integrals $\int_{\mathbb{R}^{2n}} K\varphi\, f\, d\mu$, $\int_{\mathbb{R}^{2n}} D_{x_i}\varphi K D_{x_i}\varphi\, d\mu$, and $\int_{\mathbb{R}^{2n}} D_{y_i}\varphi K D_{y_i}\varphi\, d\mu$ we get

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^{2n}} (\alpha|D_x\varphi|^2 + \beta|D_y\varphi|^2)d\mu \\
&\quad + \int_{\mathbb{R}^{2n}} (\alpha^2 \sum_{i,k}(D_{x_ix_k}\varphi)^2 + 2\alpha\beta \sum_{i,k}(D_{x_iy_k}\varphi)^2 + \beta^2 \sum_{i,k}(D_{y_iy_k}\varphi)^2)d\mu \\
&\quad + \int_{\mathbb{R}^{2n}} (\langle SD\varphi, D\varphi\rangle + 2(\langle D_yH, D_x\varphi\rangle - \langle D_xH, D_y\varphi\rangle)f\, d\mu \\
&\quad = 2\int_{\mathbb{R}^{2n}} (f - \lambda\varphi)f d\mu \le 4\|f\|^2.
\end{aligned}$$

(We have used (9) in the last inequality.) Since the second-order derivatives of H are bounded, there is $c > 0$ such that

$$\left| \int_{\mathbb{R}^{2n}} \langle SD\varphi, D\varphi\rangle d\mu \right| \le c \int_{\mathbb{R}^{2n}} |D\varphi|^2 d\mu \le \frac{2c}{\lambda}\, \|f\|^2.$$

By Lemma 2.1, the function $\langle D_yH, D_x\varphi\rangle - \langle D_xH, D_y\varphi\rangle$ belongs to $L^2(\mathbb{R}^{2n}, \mu)$, and its norm does not exceed $(C\|\,|D\varphi|^2\,\|^2 + \|\,|D^2\varphi|\,\|^2)^{1/2}$. Therefore,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{2n}} (\langle D_yH, D_x\varphi\rangle - \langle D_xH, D_y\varphi\rangle)f\, d\mu \right| \le (C\|\,|D\varphi|^2\,\|^2 + \|\,|D^2\varphi|\,\|^2)^{1/2}\|f\| \\
&\quad \le \frac{\varepsilon}{2}(C\|\,|D\varphi|^2\,\|^2 + \|\,|D^2\varphi|\,\|^2) + \frac{1}{2\varepsilon}\|f\|^2,
\end{aligned}$$

for every $\varepsilon > 0$ (we have used (10) in the last inequality). Choosing ε small enough, in such a way that

$$\varepsilon\|\,|D^2\varphi|\,\|^2 \le \int_{\mathbb{R}^{2n}} (\alpha^2 \sum_{i,k}(D_{x_ix_k}\varphi)^2 + 2\alpha\beta \sum_{i,k}(D_{x_iy_k}\varphi)^2 + \beta^2 \sum_{i,k}(D_{y_iy_k}\varphi)^2)d\mu$$

estimate (11) follows. □

Remark 2.3. In the estimate of the second-order derivatives of φ we have not used the full assumption that H has bounded second-order derivatives, but only its consequence $\langle S\xi, \xi\rangle \ge \kappa|\xi|^2$, for some $\kappa \in \mathbb{R}$ and all $\xi \in \mathbb{R}^{2n}$. This shows that the assumption that all the second-order derivatives of H could be weakened if we only want to prove that the domain of K is contained in $H^2(\mathbb{R}^{2n}, \mu)$.

3. Further properties of K and $T(t)$

The operator $\mathcal{K}$ belongs to a class of elliptic operators with unbounded coefficients recently studied in [1, 2],

$$\varphi \mapsto \mathrm{Tr}(QD^2\varphi) + \langle F, D\varphi\rangle.$$

In our case the coefficients of the matrix Q are constant, and the derivatives of every component of F are bounded. Therefore, the assumptions of [1, Prop. 4.3] are satisfied, and it yields pointwise estimates of the space derivatives of $T(t)\varphi$. Precisely,

$$|DT(t)\varphi(x,y)|^2 \le \frac{\sigma}{2\nu(1-e^{-\sigma t})}\,(T(t)\varphi^2)(x,y),\ \ \varphi\in L^2(\mathbb{R}^{2n},\mu),\ t>0,$$

where $\sigma = 2\sup_{\xi\in\mathbb{R}^{2n}\setminus\{0\}}\langle DF\cdot\xi,\xi\rangle/|\xi|^2$, DF being the Jacobian matrix of F, and ν is the ellipticity constant which is the minimum between $\alpha/2$ and $\beta/2$ in our case. Integrating over $\mathbb{R}^{2n}$ we get, for each $t>0$,

$$\|\,|DT(t)\varphi|\,\|^2_{L^2(\mathbb{R}^{2n},\mu)} \le \frac{\sigma}{2\nu(1-e^{-\sigma t})}\int_{\mathbb{R}^{2n}} T(t)\varphi^2 d\mu = \frac{\sigma}{2\nu(1-e^{-\sigma t})}\|\varphi\|^2_{L^2(\mathbb{R}^{2n},\mu)}, \tag{12}$$

which implies

$$\begin{array}{ll}\text{(i)} & \|\,|DT(t)\varphi|\,\|_{L^2(\mathbb{R}^{2n},\mu)} \le \dfrac{C}{\sqrt{t}}\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)},\ \ 0<t\le 1,\\ \text{(ii)} & \|\,|DT(t)\varphi|\,\|_{L^2(\mathbb{R}^{2n},\mu)} \le C\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)},\ \ t\ge 1.\end{array} \tag{13}$$

An important consequence of the gradient estimates (13) and of Lemma 2.1 is analyticity of $T(t)$.

Proposition 3.1. *$T(t)$ maps $L^2(\mathbb{R}^{2n},\mu)$ into $H^2(\mathbb{R}^{2n},\mu)$ for every $t>0$, and there is $C>0$ such that*

$$\|KT(t)\varphi\|_{L^2(\mathbb{R}^{2n},\mu)} \le \frac{C}{t}\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)},\ \ 0<t\le 1,\ \varphi\in L^2(\mathbb{R}^{2n},\mu).$$

Proof. Let $\varphi\in H^2(\mathbb{R}^{2n},\mu)$. Then $u(t,\cdot) := T(t)\varphi$ satisfies $u_t = \mathcal{K}u$ for $t\ge 0$. By general regularity results for parabolic equations, u_t and the second-order space derivatives of u have first-order space derivatives in $L^2_{loc}(\mathbb{R}^{2n})$, and for $i=1,\dots,n$,

$$D_{x_i}u_t = D_tD_{x_i}u = \mathcal{K}D_{x_i}u + \langle D_yD_{x_i}H, D_xu\rangle - \langle D_xD_{x_i}H, D_yu\rangle,$$

$$D_{y_i}u_t = D_tD_{y_i}u = \mathcal{K}D_{y_i}u + \langle D_yD_{y_i}H, D_xu\rangle - \langle D_xD_{y_i}H, D_yu\rangle.$$

Therefore, denoting by z_i either x_i or y_i, we have

$$D_t(D_{z_i}u - T(t)D_{z_i}\varphi) = \mathcal{K}(D_{z_i}u - T(t)D_{z_i}\varphi) + \langle D_yD_{z_i}H, D_xu\rangle - \langle D_xD_{z_i}H, D_yu\rangle$$

so that

$$\begin{aligned} &D_{z_i}T(t)\varphi - T(t)D_{z_i}\varphi\\ &\qquad = \int_0^t T(t-s)\langle D_yD_{z_i}H, D_xT(s)\varphi\rangle - \langle D_xD_{z_i}H, D_yT(s)\varphi\rangle ds := J_{z_i}\varphi(t).\end{aligned}$$

Let C be the constant in (13), and let C_1 be the maximum of the sup norm of all the second-order derivatives of H. Then for $0<t\le 1$ we have

$$\|J_{z_i}\varphi(t)\|_{L^2(\mathbb{R}^{2n},\mu)} \le C_1\int_0^t \frac{C}{\sqrt{s}}\,ds\ \|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)} = 2C_1C\sqrt{t}\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)},$$

and, more important,

$$\begin{aligned} \|\,|DJ_{z_i}\varphi(t)|\,\|_{L^2(\mathbb{R}^{2n},\mu)} &\le C_1\int_0^t \frac{C}{\sqrt{t-s}}\,\frac{C}{\sqrt{s}}\,ds\ \|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)}, \\ &= C_1C^2\int_0^1 \frac{1}{(1-\sigma)^{1/2}\sigma^{1/2}}d\sigma\ \|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)} := C_2\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)}. \end{aligned} \tag{14}$$

From the equality

$$\begin{aligned} D_{z_jz_i}T(t)\varphi &= D_{z_j}D_{z_i}T(t/2)T(t/2)\varphi \\ &= D_{z_j}T(t/2)D_{z_i}T(t/2)\varphi + D_{z_j}J_{z_i}(T(t/2)\varphi)(t/2) \end{aligned}$$

we get, using (13)(i) and (14),

$$\|D_{z_jz_i}T(t)\varphi\|_{L^2(\mathbb{R}^{2n},\mu)} \le \left(\frac{C}{\sqrt{t/2}}\right)^2 \|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)} + C_2\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)}.$$

This estimate and Lemma 2.1 yield

$$\|KT(t)\varphi\|_{L^2(\mathbb{R}^{2n},\mu)} \le \frac{C}{t}\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)}, \quad 0<t\le 1.$$

Since $H^2(\mathbb{R}^{2n},\mu)$ is dense in $L^2(\mathbb{R}^{2n},\mu)$, the statement follows. □

We can improve estimate (13)(ii), showing that $\|\,|DT(t)\varphi|\,\|_{L^2(\mathbb{R}^{2n},\mu)}$ goes to 0 as $t\to+\infty$.

Lemma 3.2. *For all $\varphi\in L^2(\mathbb{R}^{2n},\mu)$, $\lim_{t\to+\infty}\|\,|DT(t)\varphi|\,\|_{L^2(\mathbb{R}^{2n},\mu)}=0$.*

Proof. Let $\varphi\in H^2(\mathbb{R}^{2n},\mu)$. From the equality

$$\frac{d}{dt}\|T(t)\varphi\|^2 = 2\int_{\mathbb{R}^{2n}} T(t)\varphi\, KT(t)\varphi\, d\mu$$

we obtain

$$\|T(t)\varphi\|^2 - \|\varphi\|^2 = 2\int_0^t\int_{\mathbb{R}^{2n}} T(s)\varphi\, KT(s)\varphi\, d\mu\, ds$$

and using (7) we get

$$\|T(t)\varphi\|^2 + \int_0^t\int_{\mathbb{R}^{2n}} (\alpha|D_xT(s)\varphi|^2 + \beta|D_yT(s)\varphi|^2)d\mu\, ds = \|\varphi\|^2, \quad t>0. \tag{15}$$

Therefore, the function

$$\chi_\varphi(s) := \int_{\mathbb{R}^{2n}} (\alpha|D_xT(s)\varphi|^2 + \beta|D_yT(s)\varphi|^2)d\mu, \quad s\ge 0,$$

is in $L^1(0,+\infty)$, and its L^1 norm does not exceed $\|\varphi\|^2$. Its derivative is

$$\chi'_\varphi(s) = \int_{\mathbb{R}^{2n}} (2\alpha\langle D_xT(s)\varphi, D_xT(s)K\varphi\rangle + 2\beta\langle D_yT(s)\varphi, D_yT(s)K\varphi\rangle)d\mu$$

so that

$$
\begin{aligned}
|\chi'_\varphi(s)| \le\ & 2\Big(\int_{\mathbb{R}^{2n}} \alpha |D_x T(s)\varphi|^2 d\mu\Big)^{1/2}\Big(\int_{\mathbb{R}^{2n}} \alpha |D_x T(s)K\varphi|^2 d\mu\Big)^{1/2} \\
& + 2\Big(\int_{\mathbb{R}^{2n}} \beta |D_y T(s)\varphi|^2) d\mu\Big)^{1/2}\Big(\int_{\mathbb{R}^{2n}} \beta |D_y T(s)K\varphi|^2) d\mu\Big)^{1/2} \\
\le\ & \chi_\varphi(s) + \chi_{K\varphi}(s).
\end{aligned}
$$

Therefore, also χ'_φ is in $L^1(0,+\infty)$. This implies that $\lim_{s\to+\infty}\chi_\varphi(s) = 0$, and the statement holds for every $\varphi \in H^2(\mathbb{R}^{2n},\mu)$. For general $\varphi \in L^2(\mathbb{R}^{2n},\mu)$ the statement follows approaching φ by a sequence of functions in $H^2(\mathbb{R}^{2n},\mu)$ and using estimate (13)(ii). □

The lemma has some interesting consequences.

First, the kernel of K consists of constant functions. This is because if $\varphi \in \operatorname{Ker} K$ then $T(t)\varphi = \varphi$ for each $t > 0$, so that $D\varphi = DT(t)\varphi$ goes to 0 as t goes to $+\infty$, hence $D\varphi = 0$ and φ is constant. So, we have a sort of Liouville theorem for K.

Second, for each $\varphi \in L^2(\mathbb{R}^{2n},\mu)$, $T(t)\varphi$ converges weakly to the mean value $\overline{\varphi}$ of φ as $t \to +\infty$. This is because $T(t)\varphi$ is bounded, hence there is a sequence $t_n \to +\infty$ such that $T(t_n)\varphi$ converges weakly to some limit $g \in L^2(\mathbb{R}^{2n},\mu)$, each derivative $D_{z_i}T(t_n)\varphi$ converges strongly to 0, and since the derivatives are closed with respect to the weak topology too, then $D_{z_i}g = 0$ and g is constant. The constant has to be equal to the mean value of φ because μ is invariant, and hence

$$
g = \int_{\mathbb{R}^{2n}} g\,d\mu = \lim_{n\to\infty}\int_{\mathbb{R}^{2n}} T(t_n)\varphi\,d\mu = \lim_{n\to\infty}\int_{\mathbb{R}^{2n}} \varphi\,d\mu = \overline{\varphi}.
$$

Strong convergence is not obvious in general. Of course if 0 is isolated in the spectrum of K (for instance, if $H^2(\mathbb{R}^{2n},\mu)$ is compactly embedded in $L^2(\mathbb{R}^{2n},\mu)$) then $T(t)\varphi$ converges exponentially to the spectral projection of φ on the kernel of K, which is just $\overline{\varphi}$.

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Dirichlet Forms and Degenerate Elliptic Operators

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Dedicated to Philippe Clément on the occasion of his retirement

Abstract. It is shown that the theory of real symmetric second-order elliptic operators in divergence form on $\mathbb{R}^d$ can be formulated in terms of a regular strongly local Dirichlet form irregardless of the order of degeneracy. The behavior of the corresponding evolution semigroup S_t can be described in terms of a function $(A, B) \mapsto d(A\,; B) \in [0, \infty]$ over pairs of measurable subsets of $\mathbb{R}^d$. Then

$$|(\varphi_A, S_t\varphi_B)| \le e^{-d(A;B)^2(4t)^{-1}} \|\varphi_A\|_2 \|\varphi_B\|_2$$

for all $t > 0$ and all $\varphi_A \in L_2(A)$, $\varphi_B \in L_2(B)$. Moreover $S_t L_2(A) \subseteq L_2(A)$ for all $t > 0$ if and only if $d(A\,; A^c) = \infty$ where A^c denotes the complement of A.

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1. Introduction

The usual starting point for the analysis of second-order divergence-form elliptic operators with measurable coefficients is the precise definition of the operator by quadratic form techniques. Let $c_{ij} = c_{ji} \in L_\infty(\mathbb{R}^d : \mathbb{R})$, the real-valued bounded measurable functions on $\mathbb{R}^d$, and assume that the $d \times d$-matrix $C = (c_{ij})$ is positive-definite almost-everywhere. Then define the quadratic form h by

$$h(\varphi) = \sum_{i,j=1}^{d} (\partial_i \varphi, c_{ij} \partial_j \varphi) \tag{1}$$

where $\partial_i = \partial / \partial x_i$ and $\varphi \in D(h) = W^{1,2}(\mathbb{R}^d)$. It follows that h is positive and the corresponding sesquilinear form $h(\cdot\,, \cdot)$ is symmetric. Therefore if h is closed there is a canonical construction which gives a unique positive self-adjoint operator H such

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that $D(h) = D(H^{1/2})$, $h(\varphi) = \|H^{1/2}\varphi\|_2^2$ for all $\varphi \in D(h)$ and $h(\psi, \varphi) = (\psi, H\varphi)$ if $\psi \in D(h)$ and $\varphi \in D(H)$. Here and in the sequel $\|\cdot\|_p$ denotes the L_p-norm. Formally one has

$$H = -\sum_{i,j=1}^{d} \partial_i \, c_{ij} \, \partial_j \quad .$$

The critical point in the form approach is that h must be closed. But h is closed if and only if there is a $\mu > 0$ such that $C \geq \mu I$ (see, for example, [13], Proposition 2), i.e., if and only if the operator H is strongly elliptic. Therefore the construction of H is not directly applicable to degenerate elliptic operators. Nevertheless refinements of the theory of quadratic forms allow a precise definition of the elliptic operator. There are two distinct cases.

First, it is possible that h is closable on $W^{1,2}(\mathbb{R}^d)$. Then one can repeat the previous construction to obtain the self-adjoint elliptic operator associated with the closure $\overline{h}$ of h. For example, let Ω be an open subset of $\mathbb{R}^d$ with $|\partial\Omega| = 0$. Suppose $\operatorname{supp} C = \overline{\Omega}$ and $C(x) \geq \mu I > 0$ uniformly for all $x \in \Omega$. Then h is closable. Indeed the restriction of h to $W^{1,2}(\Omega)$ is closed as a form on the subspace $L_2(\Omega)$ and the corresponding self-adjoint operator H_Ω can be interpreted as the strongly elliptic operator with coefficients C acting on $L_2(\Omega)$ with Neumann boundary conditions. Then the form corresponding to the operator $H_\Omega \oplus 0$ on $L_2(\mathbb{R}^d)$, where H_Ω acts on the subspace $L_2(\Omega)$, is the closure of h.

Secondly, it is possible that h is not closable. There are many sufficient conditions for closability of the form (1) (see, for example, [14], Section 3.1, or [22], Chapter II) and if $d = 1$ then necessary and sufficient conditions are given by [14], Theorem 3.1.6. The latter conditions restrict the possible degeneracy. But Simon [30], Theorems 2.1 and 2.2, has shown that a general positive quadratic form h can be decomposed as a sum $h = h_r + h_s$ of two positive forms with $D(h_r) = D(h) = D(h_s)$ with h_r the largest closable form majorized by h. Simon refers to h_r as the regular part of h. Then one can construct the operator H_0 associated with the closure $h_0 = \overline{h_r}$ of the regular part of h. Note that if h is closed then $h_0 = h$ and if h is closable then $h_0 = \overline{h}$ so in both cases H_0 coincides with the previously defined elliptic operator. Therefore this method can be considered as the generic method of defining the operator associated with the form (1).

There is an alternative method of constructing H_0 which demonstrates more clearly its relationship with the formal definition of the elliptic operator. Introduce the form l by $D(l) = D(h) = W^{1,2}(\mathbb{R}^d)$ and

$$l(\varphi) = \sum_{i=1}^{d} \|\partial_i \varphi\|_2^2 \quad .$$

Then l is closed and the corresponding positive self-adjoint operator is the usual Laplacian Δ. Furthermore for each $\varepsilon > 0$ the form $h_\varepsilon = h + \varepsilon\, l$ with domain $D(h_\varepsilon) = D(h)$ is closed. The corresponding positive self-adjoint operators H_ε are strongly elliptic operators with coefficients $c_{ij} + \varepsilon\, \delta_{ij}$. These operators form a decreasing sequence which, by a result of Kato [21], Theorem VIII.3.11, converges in the strong resolvent sense to a positive self-adjoint operator. The limit is the operator H_0 associated with the form h_0 by [30], Theorem 3.2. The latter method of construction of H_0 as a limit of the H_ε was used in [13] and [12] and motivated the terminology viscosity operator for H_0 and viscosity form for h_0. The form h_r constructed by Simon also occurs in the context of nonlinear phenomena and discontinuous media and is variously described as the regularization or relaxation of h (see, for example, [5], [11], [20], [8], [23] and references therein).

Although the viscosity form provides a basis for the analysis of degenerate elliptic operators its indirect definition makes it difficult to deduce even the most straightforward properties. Simon, [29], Theorem 2, proved that $D(h_0)$ consists of those $\varphi \in L_2(\mathbb{R}^d)$ for which there is a sequence $\varphi_n \in D(h)$ such that $\lim_{n\to\infty} \varphi_n = \varphi$ in L_2 and $\liminf_{n\to\infty} h(\varphi_n) < \infty$. Moreover, $h_0(\varphi)$ equals the minimum of all $\liminf_{n\to\infty} h(\varphi_n)$, where the minimum is taken over all $\varphi_1, \varphi_2, \ldots \in D(h)$ such that $\lim_{n\to\infty} \varphi_n = \varphi$ in L_2. (See [29], Theorem 3.) It was observed in [13] that this characterization allows one to deduce that h_0 is a Dirichlet form. The first purpose of this note is to use the theory of Dirichlet forms (see [14], [6] for background material) to strengthen the earlier conclusion. Specifically we establish the following result in Section 2.

Theorem 1.1. *The viscosity form h_0 is a regular local Dirichlet form.*

The regularity is straightforward but the locality is not so evident and requires some control of the dissipativity of the semigroup $S^{(0)}$ generated by the viscosity operator H_0. Note that we adopt the terminology of [6] which differs from that of [14]. Locality in [6] corresponds to strong locality in [14] if the form is regular. (See [26] for a detailed study of the different forms of locality (in the sense of [14]).)

Our second purpose is to extend earlier results on L_2 off-diagonal bounds for the semigroup $S^{(0)}$ generated by the viscosity operator H_0. These bounds, which are also referred to as Davies-Gaffney estimates, integrated Gaussian estimates or an integrated maximum principle (see, for example, [2], [7], [10], [16], [31], [32]), give upper bounds on the cross-norm $\|S_t^{(0)}\|_{A\to B}$ of the semigroup $S_t^{(0)}$ between subspaces $L_2(A)$ and $L_2(B)$ of the form

$$\|S_t^{(0)}\|_{A\to B} \leq e^{-d(A\,;B)^2(4t)^{-1}} \tag{2}$$

where $d(A\,;B)$ is an appropriate measure of distance between the subsets A and B of $\mathbb{R}^d$. In the case of strongly elliptic operators there is an essentially unique distance, variously referred to as the Riemannian distance, the intrinsic distance or the control distance, suited to the problem. But for degenerate elliptic operators, which exhibit phenomena of separation and isolation [12], the Riemannian distance is not necessarily appropriate.

We will establish estimates in terms of a set-theoretic function which is not strictly a distance since it can take the value infinity. The function is defined by a version of a standard variational principle which has been used widely in the analysis of strongly elliptic operators and which was extended to the general theory of Dirichlet forms by Biroli and Mosco [3], [4]. In the case of strongly elliptic operators or subelliptic operators with smooth coefficients this method gives a distance equivalent to the Riemannian distance obtained by path methods (see, for example, [18], Section 3). In the degenerate situation we make a choice of the set of variational functions which gives a distance compatible with the separation properties. In particular $d(A\,;A^c) = \infty$ if and only if $S_t^{(0)} L_2(A) \subseteq L_2(A)$, where A^c is the complement of A. Our choice is aimed to maximize the distance and thereby optimize the bounds (2). We carry out the analysis in a general setting of local Dirichlet forms which is applicable to degenerate elliptic operators defined as above but has a wider range of applicability.

Let X be a topological Hausdorff space equipped with a σ-finite Borel measure μ. Further let $\mathcal{E}$ be a local Dirichlet form on $L_2(X)$ in the sense of [6]. First, for all $\psi \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$ define $\mathcal{I}_\psi^{(\mathcal{E})} \colon D(\mathcal{E}) \cap L_\infty(X:\mathbb{R}) \to \mathbb{R}$ by

$$\mathcal{I}_\psi^{(\mathcal{E})}(\varphi) = \mathcal{E}(\psi\,\varphi, \psi) - 2^{-1}\mathcal{E}(\psi^2, \varphi) \quad .$$

If no confusion is possible we drop the suffix and write $\mathcal{I}_\psi(\varphi) = \mathcal{I}_\psi^{(\mathcal{E})}(\varphi)$. If $\varphi \geq 0$ it follows that $\psi \mapsto \mathcal{I}_\psi(\varphi) \in \mathbb{R}$ is a Markovian form with domain $D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$ (see [6], Proposition I.4.1.1). This form is referred to as the truncated form by Roth [25], Theorem 5.

Secondly, define $D(\mathcal{E})_{\rm loc}$ as the vector space of (equivalent classes of) all measurable functions $\psi \colon X \to \mathbb{C}$ such that for every compact subset K of X there is a $\hat\psi \in D(\mathcal{E})$ with $\psi|_K = \hat\psi|_K$. Since $\mathcal{E}$ is local one can define $\widehat{\mathcal{I}}_\psi^{(\mathcal{E})} = \widehat{\mathcal{I}}_\psi \colon D(\mathcal{E}) \cap L_{\infty,c}(X:\mathbb{R}) \to \mathbb{R}$ by

$$\widehat{\mathcal{I}}_\psi(\varphi) = \mathcal{I}_{\hat\psi}(\varphi)$$

for all $\psi \in D(\mathcal{E})_{\rm loc} \cap L_\infty(X:\mathbb{R})$ and $\varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X:\mathbb{R})$ where $\hat\psi \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$ is such that $\psi|_{\operatorname{supp}\varphi} = \hat\psi|_{\operatorname{supp}\varphi}$. Here $L_{\infty,c}(X:\mathbb{R}) = \{\varphi \in L_\infty(X:\mathbb{R}) : \operatorname{supp}\varphi \text{ is compact}\}$ and $(\operatorname{supp}\varphi)^{\rm c}$ is the union of all open subsets $U \subseteq X$ such that $\varphi|_U = 0$ almost everywhere. Thirdly, for all $\psi \in D(\mathcal{E})_{\rm loc} \cap L_\infty(X:\mathbb{R})$ define

$$|||\widehat{\mathcal{I}}_\psi||| = \sup\{|\widehat{\mathcal{I}}_\psi(\varphi)| : \varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X:\mathbb{R}),\ \|\varphi\|_1 \leq 1\} \in [0,\infty] \quad .$$

Fourthly, for all $\psi \in L_\infty(X:\mathbb{R})$ and measurable sets $A, B \subset X$ introduce

$$\begin{aligned} d_\psi(A\,;B) &= \sup\{M \in \mathbb{R} : \psi(a) - \psi(b) \geq M \text{ for a.e. } a \in A \text{ and a.e. } b \in B\} \\ &= \operatorname*{ess\,inf}_{x\in A} \psi(x) - \operatorname*{ess\,sup}_{y \in B} \psi(y) \in \langle -\infty, \infty] \quad . \end{aligned}$$

(Recall that $\operatorname{ess\,sup}_{y\in B} \psi(y) = \inf\{m \in \mathbb{R} : |\{y \in B : \psi(y) > m\}| = 0\} \in [-\infty, \infty\rangle$ and $\operatorname{ess\,inf}_{x\in A} \psi(x) = -\operatorname{ess\,sup}_{x\in A} -\psi(x)$.) Finally define the set theoretic distance

$$d(A\,;B) = d^{(\mathcal{E})}(A\,;B) = \sup\{d_\psi(A\,;B) : \psi \in D_0(\mathcal{E})\} \quad ,$$

where

$$D_0(\mathcal{E}) = \{\psi \in D(\mathcal{E})_{\mathrm{loc}} \cap L_\infty(X:\mathbb{R}) : |||\widehat{\mathcal{I}}_\psi||| \leq 1\} \quad .$$

A similar definition was given by Hino and Ramirez [17], but since they consider probability spaces the introduction of $D(\mathcal{E})_{\mathrm{loc}}$ is unnecessary in [17]. If, however, we were to replace $D(\mathcal{E})_{\mathrm{loc}}$ by $D(\mathcal{E})$ in the definition of $d(A\,;B)$ then, since $\psi \in L_2(X)$, one would obtain $d_\psi(A\,;B) \leq 0$ for all measurable sets $A, B \subset X$ with $|A| = |B| = \infty$ and this would give $d(A\,;B) = 0$. On the other hand the definition with $D(\mathcal{E})_{\mathrm{loc}}$ is not useful unless $D(\mathcal{E})$ contains sufficient bounded functions of compact support.

Theorem 1.2. *Let $\mathcal{E}$ be a local Dirichlet form on $L_2(X)$ with $\mathbf{1} \in D(\mathcal{E})_{\mathrm{loc}}$ and such that $D(\mathcal{E}) \cap L_{\infty,c}(X)$ is a core for $\mathcal{E}$. Further let $d(\cdot\,;\cdot)$ denote the corresponding set-theoretic distance. If S denotes the semigroup generated by the self-adjoint operator H on $L_2(X)$ associated with $\mathcal{E}$ and if A and B are measurable subsets of X then*

$$|(\varphi_A, S_t\varphi_B)| \leq e^{-d(A\,;B)^2(4t)^{-1}} \|\varphi_A\|_2 \|\varphi_B\|_2$$

for all $\varphi_A \in L_2(A)$, $\varphi_B \in L_2(B)$ and $t > 0$ with the convention $e^{-\infty} = 0$.

This applies in particular to the viscosity operator H_0. One has $D(l) = D(h) \subseteq D(h_0)$. Hence $\mathbf{1} \in D(l)_{\mathrm{loc}} \subseteq D(h_0)_{\mathrm{loc}}$.

Theorem 1.2 will be proved in Section 3.

If h is closed, i.e., if the corresponding operator is strongly elliptic, then $d^{(h)}(\cdot\,;\cdot)$ is finite-valued. Nevertheless for degenerate operators one can have $d^{(h)}(A\,;B) = \infty$ with A, B non-empty open subsets of $\mathbb{R}^d$ and the action of the semigroup $S^{(0)}$ can be non-ergodic [12].

Specifically we establish the following result in Section 4.

Theorem 1.3. *Let $\mathcal{E}$ be a local Dirichlet form on $L_2(X)$ with $\mathbf{1} \in D(\mathcal{E})_{\mathrm{loc}}$ and such that $D(\mathcal{E}) \cap L_{\infty,c}(X)$ is a core for $\mathcal{E}$. Further let $d(\cdot\,;\cdot)$ denote the corresponding set-theoretic distance. If S denotes the semigroup generated by the self-adjoint operator H on $L_2(X)$ associated with $\mathcal{E}$ and $A \subset X$ is measurable then the following conditions are equivalent.*

I. $S_tL_2(A) \subseteq L_2(A)$ *for one* $t > 0$.
II. $S_tL_2(A) \subseteq L_2(A)$ *for all* $t > 0$.
III. $d(A\,;A^{\mathrm{c}}) = \infty$.
IV. $d(A\,;A^{\mathrm{c}}) > 0$.

We conclude by deriving alternate characterizations of the distance and deriving some of its general properties in Section 5.

2. Dirichlet forms

In this section we prove that the form h_0 defined in the introduction is a regular strongly local Dirichlet form. First we recall the basic definitions.

A Dirichlet form $\mathcal{E}$ on $L_2(X)$ is called *regular* if there is a subset of $D(\mathcal{E}) \cap C_c(X)$ which is a core of $\mathcal{E}$, i.e., which is dense in $D(\mathcal{E})$ with respect to the natural norm $\varphi \mapsto (\mathcal{E}(\varphi) + \|\varphi\|_2^2)^{1/2}$, and which is also dense in $C_0(X)$ with respect to the supremum norm. There are three kinds of locality for Dirichlet forms [6], [14]. For locality we choose the definition of [6]. A Dirichlet form $\mathcal{E}$ is called *local* if $\mathcal{E}(\psi, \varphi) = 0$ for all $\varphi, \psi \in D(\mathcal{E})$ and $a \in \mathbb{R}$ such that $(\varphi + a\mathbf{1})\psi = 0$. Alternatively, the Dirichlet form $\mathcal{E}$ is called [14]-*local* if $\mathcal{E}(\psi, \varphi) = 0$ for all $\varphi, \psi \in D(\mathcal{E})$ with $\operatorname{supp}\varphi$ and $\operatorname{supp}\psi$ compact and $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$. Moreover, it is called [14]-*strongly local* if $\mathcal{E}(\psi, \varphi) = 0$ for all $\varphi, \psi \in D(\mathcal{E})$ with $\operatorname{supp}\varphi$ and $\operatorname{supp}\psi$ compact and ψ constant on a neighborhood of $\operatorname{supp}\varphi$. Every [14]-strongly local Dirichlet form is [14]-local and if X satisfies the second axiom of countability then every local Dirichlet form (in the sense of [6]) is [14]-strongly local. If X is a locally compact separable metric space and μ is a Radon measure such that $\operatorname{supp}\mu = X$, then a regular [14]-strongly local Dirichlet form is local (in the sense of [6]) by [6] Remark I.5.1.5 and Proposition I.5.1.3 $(L_0) \Rightarrow (L_2)$.

Lemma 2.1. *The form h_0 is regular. Moreover, every core for the form l of the Laplacian is a core for h_0.*

Proof. Let h_r be the regular part of h as in [30]. Then $D(h_r) = D(h)$ and h_0 is the closure of h_r by definition. So $D(l) = D(h) = D(h_r)$ is a core for h_0. Since $h_0 \leq h \leq \|C\|\, l$, where $\|C\|$ denotes the essential supremum of the matrix norms $\|C(x)\|$, it follows that any core for l is also a core for h_0.

Finally, since $C_c^\infty(\mathbb{R}^d) \subset W^{1,2}(\mathbb{R}^d) = D(h) \subset D(h_0)$ the set $C_c^\infty(\mathbb{R}^d) \subset D(h_0) \cap C_c(\mathbb{R}^d)$ is a core of h_0 and is dense in $C_0(\mathbb{R}^d)$. □

Next we examine the locality properties.

Proposition 2.2. *The form h_0 is local.*

Proof. First note that the [14]-locality of h_0 is an easy consequence of Lemma 3.5 in [12]. Fix $\varphi, \psi \in D(h_0)$ with $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$. If D is the Euclidean distance between the support of φ and the support of ψ then

$$|h_0(\psi, \varphi)| = \lim_{t \to 0} t^{-1} |(\psi, S_t^{(0)}\varphi)| \leq \lim_{t \to 0} t^{-1} e^{-D^2 (4\|C\|t)^{-1}} \|\psi\|_2 \|\varphi\|_2 = 0$$

where we have used $(\psi, \varphi) = 0$ and the statement of [12], Lemma 3.5. The proof of [14]-strong locality is similar but depends on the estimates derived in the proof of Lemma 3.5.

Fix $\varphi, \psi \in D(h_0)$ with $\operatorname{supp}\varphi$ and $\operatorname{supp}\psi$ compact and $\psi = 1$ on a neighborhood U of $\operatorname{supp}\varphi$. It suffices, by the remark preceding Lemma 2.1, to prove that $h_0(\psi, \varphi) = 0$. We may assume the support of ψ is contained in the Euclidean ball B_R centred at the origin and with radius $R > 0$. Set $\chi_R = \mathbf{1}_{B_R}$. Since $S_t^{(0)}\mathbf{1} = \mathbf{1}$, by Proposition 3.6 of [12], one has $(\psi, \varphi) = (\mathbf{1}, \varphi) = (\mathbf{1}, S_t^{(0)}\varphi)$. Therefore

$$t^{-1}(\psi, (I - S_t^{(0)})\varphi) = t^{-1}((\chi_R - \psi), S_t^{(0)}\varphi) - t^{-1}((\chi_R - \mathbf{1}), S_t^{(0)}\varphi) \quad . \qquad (3)$$

Now let D denote the Euclidean distance from $\operatorname{supp}\varphi$ to U^{c}. It follows by assumption that $D > 0$. Then by Lemma 3.5 of [12] there is a $c > 0$ such that

$$t^{-1}|((\chi_R - \psi), S_t^{(0)}\varphi)| \le t^{-1}e^{-D^2(4\|C\|t)^{-1}}\|\chi_R - \psi\|_2\|\varphi\|_2$$
$$\le c\,R^{d/2}\,t^{-1}e^{-D^2(4\|C\|t)^{-1}} \tag{4}$$

uniformly for all large R and all $t > 0$. The factor $R^{d/2}$ comes from the L_2-norm of χ_R. Alternatively the estimate used in the proof of Proposition 3.6 in [12] establishes that there are $a, b > 0$ such that

$$|((\mathbf{1} - \chi_R), S_t^{(0)}\varphi)| \le a\sum_{n=2}^{\infty} n^{d/2}R^{d/2}e^{-bn^2R^2t^{-1}}\|\varphi\|_2$$

uniformly for all $R, t > 0$ such that $\operatorname{supp}\varphi \subset B_{2^{-1}R}$. Next there is a $c' > 0$ such that $\sum_{n=2}^{\infty} n^{d/2}\,e^{-\alpha n^2} \le c'\,\alpha^{-(d+2)/4}e^{-\alpha}$ uniformly for all $\alpha > 0$. Hence there is a $c'' > 0$ such that

$$t^{-1}|((\mathbf{1} - \chi_R), S_t^{(0)}\varphi)| \le c''\,R^{-1}t^{(d-2)/4}e^{-bR^2t^{-1}}\quad .$$

Combining this with (3) and (4) one has

$$t^{-1}|(\psi, (I - S_t^{(0)})\varphi)| \le c\,R^{d/2}t^{-1}e^{-D^2(4\|C\|t)^{-1}} + c''\,R^{-1}t^{(d-2)/4}e^{-bR^2t^{-1}}$$

for all large R and all $t > 0$. Taking the limit $t \to 0$ establishes that $h_0(\psi, \varphi) = 0$. Since h_0 is regular it follows that h_0 is local. □

3. L_2 off-diagonal bounds

In this section we prove the L_2 off-diagonal bounds of Theorem 1.2. Initially we assume that $\mathcal{E}$ is a local Dirichlet form on $L_2(X)$ without assuming any kind of regularity. The proof of the theorem follows by standard reasoning based on an exponential perturbation technique used by Gaffney [15] and subsequently developed by Davies [9] in the proof of pointwise Gaussian bounds. It is essential to establish that the perturbation of the Dirichlet form is quadratic. But this is a general consequence of locality. To exploit the latter property we use a result of Andersson [1] and [25].

Proposition 3.1. *Let $\mathcal{E}$ be a local Dirichlet form on $L_2(X)$ and $\varphi_1, \ldots, \varphi_n \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$. Then for all $i, j \in \{1, \ldots, n\}$ there exists a unique real Radon measure $\sigma_{ij}^{(\mathcal{E},\varphi_1,\ldots,\varphi_n)} = \sigma_{ij}^{(\varphi_1,\ldots,\varphi_n)}$ on $\mathbb{R}^n$ such that $\sigma_{ij}^{(\varphi_1,\ldots,\varphi_n)} = \sigma_{ji}^{(\varphi_1,\ldots,\varphi_n)}$ for all $i, j \in \{1, \ldots, n\}$ and*

$$\mathcal{E}(F_0(\varphi_1, \ldots, \varphi_n), G_0(\varphi_1, \ldots, \varphi_n)) = \sum_{i,j=1}^{n}\int_{\mathbb{R}^n} d\sigma_{ij}^{(\varphi_1,\ldots,\varphi_n)}\,\overline{\frac{\partial F}{\partial x_i}}\,\frac{\partial G}{\partial x_i} \tag{5}$$

for all $F, G \in C^1(\mathbb{R}^n)$ where $F_0 = F - F(0)$ and $G_0 = G - G(0)$. Let K be a compact subset of $\mathbb{R}^n$ such that $(\varphi_1(x), \ldots, \varphi_n(x)) \in K$ for a.e. $x \in X$. Then

$\operatorname{supp}\sigma_{ij}^{(\varphi_1,\ldots,\varphi_n)} \subseteq K$ for all $i,j \in \{1,\ldots,n\}$. In particular, if $i \in \{1,\ldots,n\}$ then $\sigma_{ii}^{(\varphi_1,\ldots,\varphi_n)}$ is a finite (positive) measure.

Moreover, if $\mathcal{F}$ is a second local Dirichlet form with $\mathcal{E} \le \mathcal{F}$ then

$$\int d\sigma_{ii}^{(\mathcal{E},\varphi_1,\ldots,\varphi_n)}\,\chi \le \int d\sigma_{ii}^{(\mathcal{F},\varphi_1,\ldots,\varphi_n)}\,\chi \tag{6}$$

for all $\varphi_1,\ldots,\varphi_n \in D(\mathcal{F}) \cap L_\infty(X:\mathbb{R})$, $i \in \{1,\ldots,n\}$ and $\chi \in C_c(\mathbb{R}^d)$ with $\chi \ge 0$.

Proof. Theorem I.5.2.1 of [6], which elaborates a result of Andersson [1], establishes that there exist unique real Radon measures $\sigma_{ij}^{(\varphi_1,\ldots,\varphi_n)}$ such that $\sigma_{ij}^{(\varphi_1,\ldots,\varphi_n)} = \sigma_{ji}^{(\varphi_1,\ldots,\varphi_n)}$ for all $i,j \in \{1,\ldots,n\}$ and (5) is valid for all $F,G \in C_c^1(\mathbb{R}^d)$. Moreover, $\sum_{i,j=1}^n \xi_i\,\xi_j\,\sigma_{ij}^{(\varphi_1,\ldots,\varphi_n)}$ is a (positive) measure for all $\xi \in \mathbb{R}^n$. Let $\xi \in \mathbb{R}^n$ and $\chi \in C_c^1(\mathbb{R}^n)$. For all $\lambda > 0$ define $F_\lambda \in C_c^1(\mathbb{R}^n)$ by $F_\lambda(x) = e^{i\lambda x\cdot\xi}\,\chi(x)$. Then it follows from (5) that

$$\lim_{\lambda\to\infty} \lambda^{-2}\,\mathcal{E}(F_{\lambda,0}(\varphi_1,\ldots,\varphi_n)) = \int \sum_{i,j=1}^n \xi_i\,\xi_j\,d\sigma_{ij}^{(\mathcal{E},\varphi_1,\ldots,\varphi_n)}\,|\chi|^2 \quad . \tag{7}$$

So if $\operatorname{supp}\chi \subset K^{\mathrm{c}}$ then the left-hand side of (7) vanishes. Hence $\operatorname{supp}\sigma_{ij}^{(\varphi_1,\ldots,\varphi_n)} \subseteq K$. But then (5) extends to all $F,G \in C^1(\mathbb{R}^n)$. Moreover the measure $\sigma_{ii}^{(\varphi_1,\ldots,\varphi_n)}$ is finite for all $i \in \{1,\ldots,n\}$ since $\sigma_{ii}^{(\varphi_1,\ldots,\varphi_n)}$ is regular.

Finally, if $\mathcal{F}$ is a second local Dirichlet form with $\mathcal{E} \le \mathcal{F}$ then it follows from (7), applied to $\mathcal{E}$ and $\mathcal{F}$, that

$$\int \sum_{i,j=1}^n \xi_i\,\xi_j\,d\sigma_{ij}^{(\mathcal{E},\varphi_1,\ldots,\varphi_n)}\,|\chi|^2 \le \int \sum_{i,j=1}^n \xi_i\,\xi_j\,d\sigma_{ij}^{(\mathcal{F},\varphi_1,\ldots,\varphi_n)}\,|\chi|^2$$

for all $\chi \in C_c^1(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$. Setting $\xi = e_i$ gives (6) by a density argument. □

Now we are prepared to prove the essential perturbation result for elements in $D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$.

Proposition 3.2. *If $\mathcal{E}$ is a local Dirichlet form then*

$$\mathcal{E}(\varphi,\varphi) - \mathcal{E}(e^{-\psi}\varphi, e^{\psi}\varphi) = \mathcal{I}_\psi^{(\mathcal{E})}(\varphi^2)$$

for all $\varphi,\psi \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$. Moreover, if $\mathcal{F}$ is a second local Dirichlet form with $\mathcal{E} \le \mathcal{F}$ then

$$\mathcal{I}_\psi^{(\mathcal{E})}(\varphi) \le \mathcal{I}_\psi^{(\mathcal{F})}(\varphi)$$

for all $\varphi,\psi \in D(\mathcal{F}) \cap L_\infty(X:\mathbb{R})$ with $\varphi \ge 0$.

Proof. Observe that

$$\begin{aligned}\mathcal{E}(\varphi,\varphi) - \mathcal{E}(e^{-\psi}\varphi, e^{\psi}\varphi) &= -\mathcal{E}((e^{-\psi}-1)\varphi, (e^{\psi}-1)\varphi) \\ &\quad - \mathcal{E}((e^{-\psi}-1)\varphi, \varphi) - \mathcal{E}(\varphi, (e^{\psi}-1)\varphi) \quad .\end{aligned}$$

Now we can apply Proposition 3.1 with $n = 2$ to each term on the right-hand side.

First, one has

$$\begin{aligned}
&-\mathcal{E}((e^{-\psi}-1)\varphi,(e^{\psi}-1)\varphi)\\
&\qquad=\int d\sigma^{(\psi,\varphi)}_{1,1}(x_1,x_2)\,x_2^2\\
&\qquad\qquad+\int d\sigma^{(\psi,\varphi)}_{1,2}(x_1,x_2)\,(e^{-x_1}(e^{x_1}-1)-e^{x_1}(e^{-x_1}-1))x_2\\
&\qquad\qquad\qquad-\int d\sigma^{(\psi,\varphi)}_{2,2}(x_1,x_2)\,(e^{-x_1}-1)(e^{x_1}-1)\quad.
\end{aligned}$$

Secondly,

$$\begin{aligned}
&-\mathcal{E}((e^{-\psi}-1)\varphi,\varphi)-\mathcal{E}(\varphi,(e^{\psi}-1)\varphi)\\
&\qquad=\int d\sigma^{(\psi,\varphi)}_{1,2}(x_1,x_2)\,(e^{-x_1}-e^{x_1})x_2\\
&\qquad\qquad-\int d\sigma^{(\psi,\varphi)}_{2,2}(x_1,x_2)\,((e^{-x_1}-1)+(e^{x_1}-1))\quad.
\end{aligned}$$

Therefore, by addition,

$$\mathcal{E}(\varphi,\varphi)-\mathcal{E}(e^{-\psi}\varphi,e^{\psi}\varphi)=\int d\sigma^{(\psi,\varphi)}_{1,1}(x_1,x_2)\,x_2^2\quad.$$

Thirdly, two more applications of Proposition 3.1 give

$$\mathcal{E}(\psi\varphi^2,\psi)=\int d\sigma^{(\psi,\varphi)}_{1,1}(x_1,x_2)\,x_2^2+2\int d\sigma^{(\psi,\varphi)}_{1,2}(x_1,x_2)\,x_1x_2$$

and

$$2^{-1}\mathcal{E}(\psi^2,\varphi^2)=2\int d\sigma^{(\psi,\varphi)}_{1,2}(x_1,x_2)\,x_1\,x_2\quad.$$

Therefore, by subtraction,

$$\mathcal{I}_\psi(\varphi^2)=\mathcal{E}(\psi\varphi^2,\psi)-2^{-1}\mathcal{E}(\psi^2,\varphi^2)=\int d\sigma^{(\psi,\varphi)}_{1,1}(x_1,x_2)\,x_2^2$$

which gives the identity in the proposition.

Finally suppose $\varphi\geq 0$. Then one calculates similarly that

$$\mathcal{I}_\psi(\varphi)=\int d\sigma^{(\psi,\varphi)}_{1,1}(x_1,x_2)\,x_2\quad.$$

But $\operatorname{supp}\sigma^{(\psi,\varphi)}_{1,1}\subseteq[-\|\psi\|_\infty,\|\psi\|_\infty]\times[0,\|\varphi\|_\infty]$ by Proposition 3.1. Hence

$$\mathcal{I}_\psi(\varphi)=\int d\sigma^{(\psi,\varphi)}_{1,1}(x_1,x_2)\,(x_2\vee 0)\quad.$$

Then the inequality follows immediately from the last part of Proposition 3.1. □

The following lemma is useful.

Lemma 3.3. *Let $\mathcal{E}$ be a local Dirichlet form.*

I. *If $\psi_1, \psi_2, \varphi \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$ with $\varphi \geq 0$ then*

$$\mathcal{I}_{\psi_1+\psi_2}(\varphi)^{1/2} \leq \mathcal{I}_{\psi_1}(\varphi)^{1/2} + \mathcal{I}_{\psi_2}(\varphi)^{1/2} \quad .$$

II. *If $\chi, \psi, \varphi \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$ then*

$$|\mathcal{I}_{\chi\,\psi}(\varphi)|^{1/2} \leq \mathcal{I}_\psi(\chi^2\,|\varphi|)^{1/2} + \mathcal{I}_\chi(\psi^2\,|\varphi|)^{1/2} \quad .$$

Proof. For all $\psi_1, \psi_2, \varphi \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$ set

$$\mathcal{I}_{\psi_1,\psi_2}(\varphi) = 2^{-1}\mathcal{E}(\psi_1\,\varphi, \psi_2) + 2^{-1}\mathcal{E}(\psi_2\,\varphi, \psi_1) - 2^{-1}\mathcal{E}(\psi_1\,\psi_2, \varphi) \quad .$$

Then

$$\Big((\psi_1, \varphi_1), (\psi_2, \varphi_2)\Big) \mapsto \mathcal{I}_{\psi_1,\psi_2}(\varphi_1\,\varphi_2)$$

is a positive symmetric bilinear form on $(D(\mathcal{E}) \cap L_\infty(X:\mathbb{R}))^2$ by [6] Proposition I.4.1.1. Hence Statement I follows by the corresponding norm triangle inequality applied to the vectors $(\psi_1, \varphi^{1/2})$ and $(\psi_2, \varphi^{1/2})$.

It follows as in the proof of Proposition 3.2 that the bilinear form satisfies the Leibniz rule

$$\mathcal{I}_{\psi_1\psi_3,\psi_2}(\varphi) = \mathcal{I}_{\psi_1,\psi_2}(\psi_3\,\varphi) + \mathcal{I}_{\psi_3,\psi_2}(\psi_1\,\varphi)$$

for all $\psi_1, \psi_2, \psi_3, \varphi \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$. But the Cauchy-Schwarz inequality states that

$$|\mathcal{I}_{\psi_1,\psi_2}(\varphi_1\,\varphi_2)| \leq \mathcal{I}_{\psi_1}(\varphi_1^2)^{1/2}\,\mathcal{I}_{\psi_2}(\varphi_2^2)^{1/2}$$

for all $\psi_1, \psi_2, \varphi_1, \varphi_2 \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$. Now let $\chi, \psi, \varphi \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$. Then

$$\begin{aligned}|\mathcal{I}_{\chi\,\psi}(\varphi)| \leq \mathcal{I}_{\chi\,\psi}(|\varphi|) &= \mathcal{I}_\chi(\psi^2\,|\varphi|) + 2\,\mathcal{I}_{\chi,\psi}(\chi\,\psi\,|\varphi|) + \mathcal{I}_\psi(\chi^2\,|\varphi|) \\ &\leq \Big(\mathcal{I}_\psi(\chi^2\,|\varphi|)^{1/2} + \mathcal{I}_\chi(\psi^2\,|\varphi|)^{1/2}\Big)^2\end{aligned}$$

and Statement II follows. □

We assume from now on in this and the next section that the Dirichlet form $\mathcal{E}$ is local, $\mathbf{1} \in D(\mathcal{E})_{\rm loc}$ and $D(\mathcal{E}) \cap L_{\infty,c}(X)$ is a core for $\mathcal{E}$. These assumptions are valid if X is a locally compact separable metric space and μ is a Radon measure such that $\operatorname{supp}\mu = X$, the Dirichlet form is regular and [14]-strongly local.

Lemma 3.4. *If $\varphi \in D(\mathcal{E})$ and $\psi \in D(\mathcal{E})_{\rm loc} \cap L_\infty(X:\mathbb{R})$ with $|||\widehat{\mathcal{I}}_\psi||| < \infty$ then $\psi\,\varphi \in D(\mathcal{E})$ and*

$$\mathcal{E}(\psi\,\varphi)^{1/2} \leq |||\widehat{\mathcal{I}}_\psi|||^{1/2}\,\|\varphi\|_2 + \|\psi\|_\infty\,\mathcal{E}(\varphi)^{1/2} \quad .$$

Proof. First assume that $\varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X:\mathbb{R})$. Since $\mathbf{1} \in D(\mathcal{E})_{\rm loc}$ there exists a $\chi \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$ such that $0 \leq \chi \leq 1$ and $\chi|_{\operatorname{supp}\varphi} = 1$. Moreover, there is a $\hat\psi \in D(\mathcal{E})$ such that $\psi|_{\operatorname{supp}\varphi} = \hat\psi|_{\operatorname{supp}\varphi}$. We may assume that $\hat\psi \in L_\infty(X:\mathbb{R})$ and $\|\hat\psi\|_\infty \leq \|\psi\|_\infty$. Then it follows from locality and Lemma 3.3.II that

$$\begin{aligned}\mathcal{E}(\psi\,\varphi)^{1/2} &= \mathcal{E}(\hat\psi\,\varphi)^{1/2} = \mathcal{I}_{\hat\psi\,\varphi}(\chi)^{1/2} \\ &\leq \mathcal{I}_{\hat\psi}(\varphi^2\,\chi)^{1/2} + \mathcal{I}_\varphi(\hat\psi^2\,\chi)^{1/2} = \widehat{\mathcal{I}}_\psi(\varphi^2\,\chi)^{1/2} + \mathcal{I}_\varphi(\hat\psi^2\,\chi)^{1/2} \\ &\leq \widehat{\mathcal{I}}_\psi(\varphi^2)^{1/2} + \|\hat\psi^2\,\chi\|_\infty^{1/2}\,\mathcal{E}(\varphi)^{1/2} \leq |||\widehat{\mathcal{I}}_\psi|||^{1/2}\,\|\varphi\|_2 + \|\psi\|_\infty\,\mathcal{E}(\varphi)^{1/2} \quad .\end{aligned}$$

Now let $\varphi \in D(\mathcal{E})$. There exists a sequence $\varphi_1, \varphi_2, \ldots \in D(\mathcal{E}) \cap L_{\infty,c}(X:\mathbb{R})$ such that $\lim_{n\to\infty} \|\varphi - \varphi_n\|_2 = 0$ and $\lim_{n\to\infty} \mathcal{E}(\varphi - \varphi_n) = 0$. Then $\lim_{n\to\infty} \psi\,\varphi_n = \psi\,\varphi$ in $L_2(X)$. Moreover, it follows from the above estimates that $n \mapsto \psi\,\varphi_n$ is a Cauchy sequence in $D(\mathcal{E})$. Since $\mathcal{E}$ is closed one deduces that $\psi\,\varphi \in D(\mathcal{E})$ and the lemma is established. □

Corollary 3.5. *If $\varphi \in D(\mathcal{E})$ and $\psi \in D(\mathcal{E})_{\rm loc} \cap L_\infty(X:\mathbb{R})$ with $|||\widehat{\mathcal{I}}_\psi||| < \infty$ then $e^\psi\varphi \in D(\mathcal{E})$ and*

$$\mathcal{E}(e^\psi\varphi)^{1/2} \leq e^{\|\psi\|_\infty}\Big(|||\widehat{\mathcal{I}}_\psi|||^{1/2}\,\|\varphi\|_2 + \mathcal{E}(\varphi)^{1/2}\Big) \quad .$$

Proof. It follows by induction from Lemma 3.4 that

$$\mathcal{E}(\psi^n\,\varphi)^{1/2} \leq n\,|||\widehat{\mathcal{I}}_\psi|||^{1/2}\,\|\psi\|_\infty^{n-1}\,\|\varphi\|_2 + \|\psi\|_\infty^n\,\mathcal{E}(\varphi)^{1/2}$$

for all $n \in \mathbb{N}$, $\varphi \in D(\mathcal{E})$ and $\psi \in D(\mathcal{E})_{\rm loc} \cap L_\infty(X:\mathbb{R})$ with $|||\widehat{\mathcal{I}}_\psi||| < \infty$. Since $\mathcal{E}(\tau)^{1/2} = \|H^{1/2}\tau\|_2$ for all $\tau \in D(\mathcal{E}) = D(H^{1/2})$ and $H^{1/2}$ is self-adjoint it follows that $e^\psi\varphi \in D(\mathcal{E})$ and

$$\begin{aligned}\mathcal{E}(e^\psi\varphi)^{1/2} &\leq \sum_{n=0}^\infty n!^{-1}\Big(n\,|||\widehat{\mathcal{I}}_\psi|||^{1/2}\,\|\psi\|_\infty^{n-1}\,\|\varphi\|_2 + \|\psi\|_\infty^n\,\mathcal{E}(\varphi)^{1/2}\Big) \\ &= e^{\|\psi\|_\infty}\Big(|||\widehat{\mathcal{I}}_\psi|||^{1/2}\,\|\varphi\|_2 + \mathcal{E}(\varphi)^{1/2}\Big)\end{aligned}$$

as required. □

Lemma 3.6. *If $\psi \in D(\mathcal{E})_{\rm loc} \cap L_\infty(X:\mathbb{R})$ with $|||\widehat{\mathcal{I}}_\psi||| < \infty$ then*

$$-\mathcal{E}(e^{-\psi}\varphi, e^\psi\varphi) \leq |||\widehat{\mathcal{I}}_\psi|||\,\|\varphi\|_2^2 \tag{8}$$

for all $\varphi \in D(\mathcal{E})$.

Proof. It follows by definition together with Proposition 3.2 that

$$\mathcal{I}_\psi(\varphi^2) = \mathcal{E}(\varphi^2\psi, \psi) - 2^{-1}\mathcal{E}(\varphi^2, \psi^2) = \mathcal{E}(\varphi) - \mathcal{E}(e^{-\psi}\varphi, e^\psi\varphi)$$

for all $\varphi, \psi \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$. Therefore

$$-\mathcal{E}(e^{-\psi}\varphi, e^\psi\varphi) = -\mathcal{E}(\varphi) + \widehat{\mathcal{I}}_\psi(\varphi^2) \leq \widehat{\mathcal{I}}_\psi(\varphi^2)$$

and (8) is valid for all $\psi \in D(\mathcal{E})_{\rm loc} \cap L_\infty(X:\mathbb{R})$ and $\varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X:\mathbb{R})$. But if $\psi \in D(\mathcal{E})_{\rm loc} \cap L_\infty(X:\mathbb{R})$ with $|||\widehat{\mathcal{I}}_\psi||| < \infty$ then the maps $\varphi \mapsto e^\psi\varphi$

and $\varphi \mapsto e^{-\psi}\varphi$ are continuous from $D(\mathcal{E})$ into $D(\mathcal{E})$ by Corollary 3.5. Moreover, $D(\mathcal{E}) \cap L_{\infty,c}(X)$ is dense in $D(\mathcal{E})$. Hence the bounds (8) extend to all $\varphi \in D(\mathcal{E})$ and $\psi \in D(\mathcal{E})_{\mathrm{loc}} \cap L_\infty(X:\mathbb{R})$ with $|||\widehat{\mathcal{I}}_\psi||| < \infty$. □

Next for all $\psi \in L_\infty(X)$ define the multiplication operator $M_\psi \colon L_2(X) \to L_2(X)$ by $M_\psi \varphi = e^\psi \varphi$.

Proposition 3.7. *If $\psi \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$ and $|||\widehat{\mathcal{I}}_\psi||| < \infty$ then*

$$\|M_\psi \, S_t \, M_\psi^{-1}\|_{2\to 2} \le e^{|||\widehat{\mathcal{I}}_\psi|||t}$$

for all $t > 0$.

Proof. Let $t > 0$. It follows from Lemma 3.6 that

$$\|e^\psi S_t\varphi\|_2^2 - \|e^\psi \varphi\|_2^2 = -2\int_0^t ds\, \mathcal{E}(S_s\varphi, e^{2\psi}S_s\varphi) \le 2\int_0^t ds\, |||\widehat{\mathcal{I}}_\psi|||\, \|e^\psi S_s\varphi\|_2^2$$

for all $\varphi \in L_2(X)$. Then it follows from Gronwall's lemma that

$$\|e^\psi S_t\varphi\|_2 \le e^{|||\widehat{\mathcal{I}}_\psi|||t}\|e^\psi\varphi\|_2$$

for all $t > 0$. □

Since $\psi \mapsto \mathcal{I}_\psi(\varphi)$ is a quadratic form one has $|||\widehat{\mathcal{I}}_{\rho\psi}||| = \rho^2|||\widehat{\mathcal{I}}_\psi|||$ for all $\rho \in \mathbb{R}$. It is now easy to complete the proof of the second theorem.

Proof of Theorem 1.2. Let $\psi \in D_0(\mathcal{E})$. Then $|||\widehat{\mathcal{I}}_\psi||| \le 1$ and

$$\begin{aligned}|(\varphi_A, S_t\varphi_B)| &\le \|e^{-\rho\psi}\varphi_A\|_2\, \|M_{\rho\psi}\, S_t\, M_{\rho\psi}^{-1}\|_{2\to 2}\, \|e^{\rho\psi}\varphi_B\|_2 \\ &\le e^{|||\widehat{\mathcal{I}}_{\rho\psi}|||t}\, e^{-d_{\rho\psi}(A;B)}\, \|\varphi_A\|_2\, \|\varphi_B\|_2 \le e^{\rho^2 t}\, e^{-\rho d_\psi(A;B)}\, \|\varphi_A\|_2\, \|\varphi_B\|_2\end{aligned}$$

for all $\rho > 0$. Minimizing over ψ gives

$$|(\varphi_A, S_t\varphi_B)| \le e^{\rho^2 t}\, e^{-\rho d(A;B)}\, \|\varphi_A\|_2\, \|\varphi_B\|_2$$

for all $\rho > 0$. If $d(A\,;B) = \infty$ then $(\varphi_A, S_t\varphi_B) = 0$ and the theorem follows. Finally, if $d(A\,;B) < \infty$ choose $\rho = (2t)^{-1}d(A\,;B)$. □

One immediate corollary of the theorem is that the corresponding wave equation has a finite speed of propagation by the reasoning of [27].

Corollary 3.8. *Let H be the positive self-adjoint operator associated with a local Dirichlet form $\mathcal{E}$ on $L_2(X)$ such that $\mathbf{1} \in D(\mathcal{E})_{\mathrm{loc}}$ and $D(\mathcal{E}) \cap L_{\infty,c}(X)$ is a core for $\mathcal{E}$. If A, B are measurable subsets of X then*

$$(\varphi_A, \cos(tH^{1/2})\varphi_B) = 0$$

for all $\varphi_A \in L_2(A)$, $\varphi_B \in L_2(B)$ and $t \in [-d(A\,;B), d(A\,;B)]$.

Proof. This follows immediately from Theorem 1.2 and Lemma 3.3 of [12]. □

We have already remarked in the introduction that Theorem 1.2 applies directly to the second-order viscosity operators. Alternatively one may deduce off-diagonal bounds for operators on open subsets of $\mathbb{R}^d$ satisfying Dirichlet or Neumann boundary conditions. As an illustration consider the Laplacian.

Example 3.9. Let X be an open subset of $\mathbb{R}^d$. Define $\mathcal{E}$ on $L_2(X)$ by $D(\mathcal{E}) = W_0^{1,2}(X)$ and $\mathcal{E}(\varphi) = \|\nabla\varphi\|_2^2$. In particular $D(\mathcal{E})$ is the closure of $C_c^\infty(X)$ with respect to the norm $\varphi \mapsto (\|\nabla\varphi\|_2^2 + \|\varphi\|_2^2)^{1/2}$. It follows that $\mathcal{E}$ is a regular local Dirichlet form and the assumptions of Theorem 1.2 are satisfied. Therefore Theorem 1.2 gives off-diagonal bounds. The self-adjoint operator corresponding to $\mathcal{E}$ is the Dirichlet Laplacian on $L_2(X)$.

Example 3.10. Let Ω be an open subset of $\mathbb{R}^d$ with $|\partial\Omega| = 0$. Set $X = \overline{\Omega}$. Define the form $\mathcal{E}$ on $L_2(X) = L_2(\Omega)$ by $D(\mathcal{E}) = W^{1,2}(\Omega)$ and $\mathcal{E}(\varphi) = \|\nabla\varphi\|_2^2$. Then $\mathcal{E}$ is a local Dirichlet form. Fix $\tau \in C_c^\infty(\mathbb{R}^d)$ with $0 \leq \tau \leq 1$ and $\tau|_{B_e(0;1)} = 1$, where $B_e(0\,;1)$ is the Euclidean unit ball. For all $n \in \mathbb{N}$ define $\tau_n \in C_c^\infty(\mathbb{R}^d)$ by $\tau_n(x) = \tau(n^{-1}x)$. Then $\tau_n|_\Omega \in D(\mathcal{E})$ for all $n \in \mathbb{N}$ and therefore $\mathbf{1} \in D(\mathcal{E})_{\rm loc}$. The space $D(\mathcal{E}) \cap L_\infty(X)$ is dense in $D(\mathcal{E})$ by [14] Theorem 1.4.2.(iii). If $\varphi \in D(\mathcal{E}) \cap L_\infty(X)$ then $\lim_{n\to\infty} \tau_n|_\Omega\, \varphi = \varphi$ in $W^{1,2}(\Omega) = D(\mathcal{E})$. But for all $n \in \mathbb{N}$ the support of the function $\tau_n|_\Omega\, \varphi$, viewed as an almost everywhere defined function on X, is closed in X, and therefore also closed in $\mathbb{R}^d$. Hence this support is compact in $\mathbb{R}^d$ and then also compact in X. So $\tau_n|_\Omega\, \varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X)$ and the space $D(\mathcal{E}) \cap L_{\infty,c}(X)$ is dense in $D(\mathcal{E})$. Therefore Theorem 1.2 gives off-diagonal bounds. The self-adjoint operator corresponding to $\mathcal{E}$ is the Neumann Laplacian on $L_2(\Omega)$.

Note that we do not assume that Ω has the segment property. In general the Dirichlet form $\mathcal{E}$ is not regular on X.

4. Separation

In this section we give the proof of Theorem 1.3. In [12] we established that for degenerate elliptic operators phenomena of separation can occur. In particular the corresponding semigroup S is reducible, i.e., it has non-trivial invariant subspaces. The theorem shows that such subspaces can be characterized by the set-theoretic distance $d(\cdot\,;\cdot)$.

For the proof of the implication I⇒II in Theorem 1.3 we need a variation of an argument used to prove Theorem XIII.44 in [24] (see also [28]). The essence of the argument is contained in the following lemma.

Lemma 4.1. *Let A,B be measurable subsets of X with finite measure. If $(S_t\mathbf{1}_A,\mathbf{1}_B) = 0$ for one $t > 0$ then $(S_t\mathbf{1}_A, \mathbf{1}_B) = 0$ for all $t > 0$.*

Proof. Set $P = \{t \in \langle 0,\infty\rangle : (S_t\mathbf{1}_A, \mathbf{1}_B) > 0\}$. Let $t \in P$ and $\tau > 0$. Since $(S_t\mathbf{1}_A, \mathbf{1}_B) > 0$ it follows that the product $(S_t\mathbf{1}_A) \cdot (\mathbf{1}_B)$ is not identically zero. Let $\varphi = S_t\mathbf{1}_A \wedge \mathbf{1}_B$. Then $\varphi \neq 0$. Moreover,

$$(S_{t+\tau}\mathbf{1}_A, \mathbf{1}_B) = (S_\tau(S_t\mathbf{1}_A), \mathbf{1}_B) \geq (S_\tau\varphi, \varphi) = \|S_{\tau/2}\varphi\|_2^2 > 0$$

because S is Markovian. Thus $t + \tau \in P$ and $[t,\infty\rangle \subset P$ for all $t \in P$.

Set $N = \langle 0,\infty\rangle \backslash P = \{t \in \langle 0,\infty\rangle : (S_t \mathbf{1}_A, \mathbf{1}_B) = 0\}$. If $N \neq \emptyset$ then there is an $a > 0$ such that $\langle 0, a\rangle \subset N$. But the map $z \mapsto (S_z \mathbf{1}_A, \mathbf{1}_B)$ is analytic in the open right half-plane. Hence if $N \neq \emptyset$ then $(S_t \mathbf{1}_A, \mathbf{1}_B) = 0$ for all $t \in \langle 0,\infty\rangle$. □

Before we prove Theorem 1.3 we need one more lemma. Recall that we assume throughout this section that the Dirichlet form $\mathcal{E}$ is local, $\mathbf{1} \in D(\mathcal{E})_{\mathrm{loc}}$ and $D(\mathcal{E}) \cap L_{\infty,c}(X)$ is a core for $\mathcal{E}$. A normal contraction on $\mathbb{R}$ is a function $F\colon \mathbb{R} \to \mathbb{R}$ such that $|F(x) - F(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$ and $F(0) = 0$.

Lemma 4.2.

I. *If $\psi \in D(\mathcal{E})_{\mathrm{loc}} \cap L_\infty(X:\mathbb{R})$ and $\lambda \in \mathbb{R}$ then $\psi + \lambda\mathbf{1} \in D(\mathcal{E})_{\mathrm{loc}} \cap L_\infty(X:\mathbb{R})$ and $\widehat{\mathcal{I}}_{\psi+\lambda\mathbf{1}} = \widehat{\mathcal{I}}_\psi$. In particular $|||\widehat{\mathcal{I}}_{\psi+\lambda\mathbf{1}}||| = |||\widehat{\mathcal{I}}_\psi|||$.*

II. *If $\psi \in D(\mathcal{E})_{\mathrm{loc}} \cap L_\infty(X:\mathbb{R})$ and F is a normal contraction then $F \circ \psi \in D(\mathcal{E})_{\mathrm{loc}} \cap L_\infty(X:\mathbb{R})$ and $|||\widehat{\mathcal{I}}_{F\circ\psi}||| \leq |||\widehat{\mathcal{I}}_\psi|||$.*

III. *If A, B are measurable subsets of X, $M \in [0, d(A;B)] \cap \mathbb{R}$ and $\varepsilon > 0$ then there is a $\psi \in D_0(\mathcal{E})$ such that $0 \leq \psi \leq M$, $\psi(b) = 0$ for a.e. $b \in B$ and $\psi(a) \geq M - \varepsilon$ for a.e. $a \in A$.*

Proof. Let $\varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X:\mathbb{R})$. There exist $\hat\psi, \chi \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$ such that $\hat\psi|_{\operatorname{supp}\varphi} = \psi|_{\operatorname{supp}\varphi}$ and $\chi|_{\operatorname{supp}\varphi} = 1$. Then the locality of $\mathcal{E}$ implies that

$$\begin{aligned}\widehat{\mathcal{I}}_{\psi+\lambda\mathbf{1}}(\varphi) &= \mathcal{I}_{\hat\psi+\lambda\chi}(\varphi) = \mathcal{E}((\hat\psi+\lambda\chi)\varphi, \hat\psi+\lambda\chi) - 2^{-1}\mathcal{E}(\varphi, (\hat\psi+\lambda\chi)^2) \\ &= \mathcal{E}(\hat\psi\,\varphi, \hat\psi) + \lambda\,\mathcal{E}(\varphi, \hat\psi) - 2^{-1}\Big(\mathcal{E}(\varphi, \hat\psi^2) - 2\mathcal{E}(\varphi, \lambda\,\hat\psi\,\chi)\Big) \\ &= \mathcal{E}(\hat\psi\,\varphi, \hat\psi) - 2^{-1}\mathcal{E}(\varphi, \hat\psi^2) = \mathcal{I}_{\hat\psi}(\varphi) = \widehat{\mathcal{I}}_\psi(\varphi) \quad .\end{aligned}$$

This proves Statement I.

If F is a normal contraction on $\mathbb{R}$ then $\mathcal{I}_{F\circ\psi}(\varphi) \leq \mathcal{I}_\psi(\varphi)$ for all $\varphi, \psi \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$ with $\varphi \geq 0$ by [6], Proposition I.4.1.1. Then Statement II is an easy consequence.

Finally, there exists a $\psi \in D_0(\mathcal{E})$ such that $d_\psi(A;B) \geq M - \varepsilon$. We may assume that $|B| > 0$. Then the truncation $0 \vee (\psi + \|\mathbf{1}_B\,\psi\|_\infty) \wedge M$ satisfies the required conditions. □

Proof of Theorem 1.3. The implication I⇒II is now immediate. Let $B \subset A^{\mathrm{c}}$, $A' \subset A$ and suppose that A' and B have finite measure. Then $(S_t \mathbf{1}_{A'}, \mathbf{1}_B) = 0$ for one $t > 0$ by assumption and for all $t > 0$ by Lemma 4.1. Hence $(S_t\varphi, \psi) = 0$ for all $t > 0$, $\varphi \in L_2(A)$ and $\psi \in L_2(A^{\mathrm{c}})$. Thus $S_t L_2(A) \subseteq L_2(A)$ for all $t > 0$.

The converse implication II⇒I is obvious.

Next we prove the implication II⇒III. For all $\varphi\colon X \to \mathbb{C}$ set $\widetilde\varphi = \mathbf{1}_A\,\varphi$. Then $\widetilde\varphi \in D(\mathcal{E})$ and $\mathcal{E}(\psi, \widetilde\varphi) = \mathcal{E}(\widetilde\psi, \varphi) = \mathcal{E}(\widetilde\psi, \widetilde\varphi)$ for all $\varphi, \psi \in D(\mathcal{E})$ by [12], Lemma 6.3. Hence

$$\mathcal{I}_{\widetilde\psi}(\varphi) = \mathcal{E}(\widetilde\psi\,\varphi, \widetilde\psi) - 2^{-1}\mathcal{E}(\varphi, \widetilde{\psi}^2) = \mathcal{I}_\psi(\widetilde\varphi)$$

for all $\varphi, \psi \in D(\mathcal{E}) \cap L_\infty(X : \mathbb{R})$. Therefore, if $\psi \in D(\mathcal{E})_{\rm loc} \cap L_\infty(X : \mathbb{R})$ and $\varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X : \mathbb{R})$ then $\tilde{\psi} \in D(\mathcal{E})_{\rm loc} \cap L_\infty(X : \mathbb{R})$ and $\widehat{\mathcal{I}}_{\tilde{\psi}}(\varphi) = \widehat{\mathcal{I}}_\psi(\tilde{\varphi})$. But $\widehat{\mathcal{I}}_{\mathbf{1}} = 0$ by Lemma 4.2.I since $\mathcal{E}$ is local. So $\widehat{\mathcal{I}}_{\mathbf{1}_A}(\varphi) = \widehat{\mathcal{I}}_{\tilde{\mathbf{1}}}(\varphi) = \widehat{\mathcal{I}}_{\mathbf{1}}(\tilde{\varphi}) = 0$ for all $\varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X : \mathbb{R})$. So $|||\widehat{\mathcal{I}}_{\mathbf{1}_A}||| = 0$. Then $d(A\,;A^{\rm c}) = \infty$.

The implication III$\Rightarrow$IV is trivial.

Finally suppose that IV is valid. Let $\delta \in \langle 0, \infty\rangle$ such that $2\delta < d(A\,;A^{\rm c})$. By Lemma 4.2.III there exists a $\psi \in D(\mathcal{E})_{\rm loc} \cap L_\infty(X : \mathbb{R})$ such that $\psi(b) = 0$ for a.e. $b \in A^{\rm c}$ and $\psi(a) \geq \delta$ for a.e. $a \in A$. Then $\mathbf{1}_A = 0 \vee (\delta^{-1}\psi_2) \wedge 1$ and since $x \mapsto 0 \vee x \wedge 1$ is a normal contraction it follows from Lemma 4.2.II that $\mathbf{1}_A \in D(\mathcal{E})_{\rm loc} \cap L_\infty(X : \mathbb{R})$ and $|||\widehat{\mathcal{I}}_{\mathbf{1}_A}||| \leq \delta^{-2} < \infty$.

Now let $\varphi \in D(\mathcal{E})$. It follows from Lemma 3.4 that $\mathbf{1}_A\,\varphi \in D(\mathcal{E})$. Then also $\mathbf{1}_{A^{\rm c}}\,\varphi \in D(\mathcal{E})$. For all $t > 0$ one has by Theorem 1.2 that

$$\begin{aligned} t^{-1}\,|(\mathbf{1}_A\,\varphi, (I - S_t)(\mathbf{1}_{A^{\rm c}}\,\varphi))| &= t^{-1}\,|(\mathbf{1}_A\,\varphi, S_t(\mathbf{1}_{A^{\rm c}}\,\varphi))| \\ &\leq t^{-1}\,e^{-\delta^2 t^{-1}}\,\|\mathbf{1}_A\,\varphi\|_2\,\|\mathbf{1}_{A^{\rm c}}\,\varphi\|_2 \quad . \end{aligned}$$

Hence

$$\mathcal{E}(\mathbf{1}_A\,\varphi, \mathbf{1}_{A^{\rm c}}\,\varphi) = \lim_{t\downarrow 0} t^{-1}\,(\mathbf{1}_A\,\varphi, (I - S_t)(\mathbf{1}_{A^{\rm c}}\,\varphi)) = 0$$

and

$$\mathcal{E}(\varphi) = \mathcal{E}(\mathbf{1}_A\,\varphi) + \mathcal{E}(\mathbf{1}_{A^{\rm c}}\,\varphi) \quad .$$

Then [12], Lemma 6.3, implies that II is valid. □

Note that the above proof shows that the equivalent statements of Theorem 1.3 are also equivalent with the statement $\mathbf{1}_A \in D(\mathcal{E})_{\rm loc} \cap L_\infty(X : \mathbb{R})$ and $|||\widehat{\mathcal{I}}_{\mathbf{1}_A}||| < \infty$.

Example 4.3. Let $\delta \in [0, 1\rangle$ and consider the viscosity form h on $\mathbb{R}^d$ with $d = 1$ and coefficient $c_{11} = c_\delta$ where

$$c_\delta(x) = \Big(\frac{x^2}{1 + x^2}\Big)^\delta \quad .$$

Then $h(\varphi) = \int |\varphi'(x)|^2\,c_\delta(x)$ for all $\varphi \in C_c^\infty(\mathbb{R})$, the form h is closable and its closure is a Dirichlet form. If S is the semigroup generated by the operator associated with the closure and if $\delta \in [1/2, 1\rangle$ then $S_t L_2(-\infty, 0) \subseteq L_2(-\infty, 0)$ and $S_t L_2(0, \infty) \subseteq L_2(0, \infty)$ for all $t > 0$ by [12], Corollary 2.4 and Proposition 6.5. The assumptions of Theorems 1.2 and 1.3 are satisfied and $d(A\,;B) = \infty$ for each pair of measurable subsets $A \subseteq \langle -\infty, 0\rangle$ and $B \subseteq \langle 0, \infty\rangle$. Note, however, that the corresponding Riemannian distance

$$d(x\,;y) = \Big|\int_y^x ds\,c_\delta(s)^{-1/2}\Big|$$

is finite for all $x, y \in \mathbb{R}$. Therefore the Riemannian distance does not reflect the behavior of the semigroup.

5. Distances

In this section we derive various general properties of the distance $d^{(\mathcal{E})}(\cdot\,;\cdot)$ and give several examples.

Let $\mathcal{E}$ be a local Dirichlet form on $L_2(X)$. If $A_1 \subseteq A_2$ and B are measurable then $d_\psi(A_1\,;B) \geq d_\psi(A_2\,;B)$ for all $\psi \in L_\infty(X)$. Hence $d(A_1\,;B) \geq d(A_2\,;B)$. Also $d_\psi(B\,;A) = d_{-\psi}(A\,;B)$ for all $\psi \in L_\infty(X)$. So $d(A\,;B) = d(B\,;A)$ and $d(A\,;B_1) \geq d(A\,;B_2)$ whenever $B_1 \subseteq B_2$ are measurable. Next we consider monotonicity of the distance as a function of the form.

If h_1 and h_2 are two strongly elliptic forms on $\mathbb{R}^d$ and $h_1 \geq h_2$ then the corresponding matrices of coefficients satisfy $C^{(1)} \geq C^{(2)}$. Therefore $(C^{(1)})^{-1} \leq (C^{(2)})^{-1}$. Hence the corresponding Riemannian distances satisfy $d_1(x\,;y) \leq d_2(x\,;y)$ for all $x, y \in \mathbb{R}^d$. Thus the order of the forms gives the inverse order for the distances. One can establish a similar result for general Dirichlet forms and the set-theoretic distance under subsidiary regularity conditions.

Proposition 5.1. *Let $\mathcal{E}$ and $\mathcal{F}$ be local Dirichlet forms with $\mathcal{E} \leq \mathcal{F}$. Assume $\mathbf{1} \in D(\mathcal{F})_{\rm loc}$, the space $D(\mathcal{F}) \cap L_{\infty,c}(X)$ is a core for $D(\mathcal{E})$ and for each compact $K \subset X$ there is a $\chi \in D(\mathcal{F}) \cap L_{\infty,c}(X:\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi|_K = 1$ and $|||\widehat{I}^{(\mathcal{F})}_\chi||| < \infty$. Then*

$$d^{(\mathcal{F})}(A\,;B) \leq d^{(\mathcal{E})}(A\,;B)$$

for all measurable $A, B \subseteq X$.

Proof. It suffices to prove that $|||\widehat{\mathcal{I}}^{(\mathcal{E})}_\psi||| \leq |||\widehat{\mathcal{I}}^{(\mathcal{F})}_\psi|||$ for all $\psi \in D(\mathcal{F})_{\rm loc} \cap L_\infty(X:\mathbb{R})$ because the statement of the proposition then follows from the definition of the distance.

If $\psi, \varphi \in D(\mathcal{F}) \cap L_\infty(X:\mathbb{R})$ with $\varphi \geq 0$ then $\mathcal{I}^{(\mathcal{E})}_\psi(\varphi) \leq \mathcal{I}^{(\mathcal{F})}_\psi(\varphi)$ by Proposition 3.2. Therefore $\widehat{\mathcal{I}}^{(\mathcal{E})}_\psi(\varphi) \leq \widehat{\mathcal{I}}^{(\mathcal{F})}_\psi(\varphi)$ for all $\psi \in D(\mathcal{F})_{\rm loc} \cap L_\infty(X:\mathbb{R})$ and $\varphi \in D(\mathcal{F}) \cap L_{\infty,c}(X:\mathbb{R})$ with $\varphi \geq 0$.

Fix $\psi \in D(\mathcal{F})_{\rm loc} \cap L_\infty(X:\mathbb{R})$. If $|||\widehat{\mathcal{I}}^{(\mathcal{F})}_\psi||| = \infty$ there is nothing to prove, so we may assume that $|||\widehat{\mathcal{I}}^{(\mathcal{F})}_\psi||| < \infty$. It follows as in the proof of Lemma 3.4 that

$$\begin{aligned}\mathcal{E}(\psi\varphi)^{1/2} &\leq \widehat{\mathcal{I}}^{(\mathcal{E})}_\psi(\varphi^2)^{1/2} + \|\psi\|_\infty\,\mathcal{E}(\varphi)^{1/2}\\ &\leq \widehat{\mathcal{I}}^{(\mathcal{F})}_\psi(\varphi^2)^{1/2} + \|\psi\|_\infty\,\mathcal{E}(\varphi)^{1/2} \leq |||\widehat{\mathcal{I}}^{(\mathcal{F})}_\psi|||\,\|\varphi\|_2 + \|\psi\|_\infty\,\mathcal{E}(\varphi)^{1/2}\end{aligned}$$

for all $\varphi \in D(\mathcal{F}) \cap L_{\infty,c}(X:\mathbb{R})$. Since $\mathcal{E}$ is closed and $D(\mathcal{F}) \cap L_{\infty,c}(X)$ is a core it then follows that $\psi\varphi \in D(\mathcal{E})$ for all $\varphi \in D(\mathcal{E})$ and

$$\mathcal{E}(\psi\varphi)^{1/2} \leq |||\widehat{\mathcal{I}}^{(\mathcal{F})}_\psi|||\,\|\varphi\|_2 + \|\psi\|_\infty\,\mathcal{E}(\varphi)^{1/2}$$

for all $\varphi \in D(\mathcal{E})$.

Let $\varphi_0 \in D(\mathcal{E}) \cap L_{\infty,c}(X:\mathbb{R})$. By assumption there exists a $\chi \in D(\mathcal{F}) \cap L_{\infty,c}(X:\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi|_{\operatorname{supp}\varphi_0} = 1$ and $|||\widehat{\mathcal{I}}^{(\mathcal{F})}_\chi||| < \infty$. Then by the above argument with ψ replaced by χ one deduces that

$$\mathcal{E}(\chi\varphi)^{1/2} \leq |||\widehat{\mathcal{I}}^{(\mathcal{F})}_\chi||| \, \|\varphi\|_2 + \mathcal{E}(\varphi)^{1/2}$$

for all $\varphi \in D(\mathcal{E})$. There exists a $\hat\psi \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$ such that $\hat\psi|_{\operatorname{supp}\chi} = \psi|_{\operatorname{supp}\chi}$. Then there is a $c > 0$ such that

$$\begin{aligned}
|\widehat{\mathcal{I}}^{(\mathcal{E})}_\psi(\chi\,\varphi)| = |\mathcal{I}^{(\mathcal{E})}_{\hat\psi}(\chi\,\varphi)| &\leq \mathcal{E}(\hat\psi)^{1/2}\mathcal{E}(\hat\psi\,\chi\,\varphi)^{1/2} + 2^{-1}\mathcal{E}(\hat\psi^2)^{1/2}\,\mathcal{E}(\chi\,\varphi)^{1/2}\\
&= \mathcal{E}(\hat\psi)^{1/2}\mathcal{E}(\psi\,\chi\,\varphi)^{1/2} + 2^{-1}\mathcal{E}(\hat\psi^2)^{1/2}\,\mathcal{E}(\chi\,\varphi)^{1/2}\\
&\leq \mathcal{E}(\hat\psi)^{1/2}\Big(|||\widehat{\mathcal{I}}^{(\mathcal{F})}_\psi|||\,\|\chi\,\varphi\|_2 + \|\psi\|_\infty\,\mathcal{E}(\chi\,\varphi)^{1/2}\Big) + 2^{-1}\mathcal{E}(\psi^2)^{1/2}\,\mathcal{E}(\chi\,\varphi)^{1/2}\\
&\leq c\Big(\mathcal{E}(\varphi)^{1/2} + \|\varphi\|_2\Big)
\end{aligned}$$

uniformly for all $\varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X:\mathbb{R})$. Since the space $D(\mathcal{F}) \cap L_{\infty,c}(X:\mathbb{R})$ is dense in the space $D(\mathcal{E})$ there are $\varphi_1, \varphi_2, \ldots \in D(\mathcal{F}) \cap L_{\infty,c}(X:\mathbb{R})$ such that $\lim_{n\to\infty}(\|\varphi_0 - \varphi_n\|_2 + \mathcal{E}(\varphi_0 - \varphi_n)) = 0$. Then

$$\lim_{n\to\infty} \widehat{\mathcal{I}}^{(\mathcal{E})}_\psi(\chi\varphi_n) = \widehat{\mathcal{I}}^{(\mathcal{E})}_\psi(\chi\varphi_0) = \widehat{\mathcal{I}}^{(\mathcal{E})}_\psi(\varphi_0) \quad .$$

On the other hand,

$$|\widehat{\mathcal{I}}^{(\mathcal{E})}_\psi(\chi\varphi_n)| \leq |||\widehat{\mathcal{I}}^{(\mathcal{F})}_\psi|||\,\|\chi\varphi_n\|_1 \leq |||\widehat{\mathcal{I}}^{(\mathcal{F})}_\psi|||\,\|\varphi_0\|_1 + |||\widehat{\mathcal{I}}^{(\mathcal{F})}_\psi|||\,\|\chi\|_2\,\|\varphi_0 - \varphi_n\|_2$$

for all $n \in \mathbb{N}$. Taking the limit $n \to \infty$ gives $|\widehat{\mathcal{I}}^{(\mathcal{E})}_\psi(\varphi_0)| \leq |||\widehat{\mathcal{I}}^{(\mathcal{F})}_\psi|||\,\|\varphi_0\|_1$. The required monotonicity of the norms is immediate. □

One can apply the proposition to a general elliptic form h as in (1) to obtain a lower bound on the distance. Then the viscosity form h_0 satisfies $h_0 \leq h \leq \|C\|\,l$ where l is the form associated with the Laplacian. Therefore

$$d^{(h_0)}(A\,;B) \geq \|C\|^{-1/2} d^{(l)}(A\,;B)$$

for all measurable $A, B \subseteq \mathbb{R}^d$. Alternatively if h is a strongly elliptic form then there is a $\mu > 0$ such that $C \geq \mu I$ and $\mu\,l \leq h \leq \|C\|\,l$. It then follows that $d^{(h)}(\cdot\,;\cdot)$ is equivalent to the distance $d^{(l)}(\cdot\,;\cdot)$. Specifically,

$$\|C\|^{-1/2} d^{(l)}(A\,;B) \leq d^{(h)}(A\,;B) \leq \mu^{-1/2} d^{(l)}(A\,;B)$$

for all measurable sets A and B.

In the case of the Laplacian one can explicitly identify the distance.

Example 5.2. If A and B are non-empty open subsets of $\mathbb{R}^d$ then

$$d^{(l)}(A\,;B) = \inf_{x\in A}\,\inf_{y\in B}|x - y| \quad . \tag{9}$$

First, observe that if $\varphi, \psi \in D(l) \cap L_\infty(\mathbb{R}^d:\mathbb{R}) = W^{1,2} \cap L_\infty(\mathbb{R}^d:\mathbb{R})$ then

$$\mathcal{I}^{(l)}_\psi(\varphi) = (\nabla(\varphi\psi), \nabla\psi) - 2^{-1}(\nabla\varphi, \nabla(\psi^2)) = \int_{\mathbb{R}^d} dx\,\varphi(x)\,|(\nabla\psi)(x)|^2 \quad .$$

Secondly, let $\psi \in W^{1,2}_{\rm loc} \cap L_\infty(\mathbb{R}^d:\mathbb{R})$ and $K \subset \mathbb{R}^d$ compact. Choose $\hat\psi \in W^{1,2} \cap L_\infty$ such that $\psi|_K = \hat\psi|_K$. If $|||\widehat{\mathcal{I}}^{(l)}_\psi||| \leq 1$ then

$$\Big|\int_K dx\,\varphi(x)\,|(\nabla\psi)(x)|^2\Big| = \Big|\int_K dx\,\varphi(x)\,|(\nabla\hat\psi)(x)|^2\Big| = |\mathcal{I}^{(l)}_{\hat\psi}(\varphi)| = |\widehat{\mathcal{I}}^{(l)}_\psi(\varphi)| \leq \|\varphi\|_1$$

for all $\varphi \in W^{1,2} \cap L_\infty(\mathbb{R}^d:\mathbb{R}) \cap L_1$ with $\operatorname{supp}\varphi \subseteq K$.

Therefore one has $\sup_{x\in K}|(\nabla\psi)(x)| \leq 1$ uniformly for all K and $\psi \in W^{1,\infty}$. Thus $D_0(l) = \{\psi \in W^{1,\infty}(\mathbb{R}^d:\mathbb{R}) : \|\nabla\psi\|_\infty \leq 1\}$. But it is well known that

$$|x-y| = \sup\{|\psi(x)-\psi(y)| : W^{1,\infty}(\mathbb{R}^d:\mathbb{R})\,,\ \|\nabla\psi\|_\infty \leq 1\}$$

(see, for example, [19], Proposition 3.1). Then (9) follows immediately.

The next proposition compares forms under a rather different regularity assumption. We define $D(\mathcal{F})$ to be an *ideal* of $D(\mathcal{E})$ if $D(\mathcal{F}) \subseteq D(\mathcal{E})$ and

$$\Big(D(\mathcal{F})_{\rm loc} \cap L_\infty(X:\mathbb{R})\Big)\Big(D(\mathcal{E}) \cap L_{\infty;c}(X:\mathbb{R})\Big) \subseteq \Big(D(\mathcal{F})_{\rm loc} \cap L_\infty(X:\mathbb{R})\Big)\quad.$$

Note that if $\mathbf{1} \in D(\mathcal{F})_{\rm loc}$ and if $D(\mathcal{F})$ is an ideal of $D(\mathcal{E})$ then

$$\psi = \mathbf{1}\,\psi \in \Big(D(\mathcal{F})_{\rm loc}\cap L_\infty(X:\mathbb{R})\Big)\Big(D(\mathcal{E})\cap L_{\infty;c}(X:\mathbb{R})\Big) \subseteq \Big(D(\mathcal{F})_{\rm loc}\cap L_{\infty;c}(X:\mathbb{R})\Big)$$

for all $\psi \in D(\mathcal{E}) \cap L_{\infty;c}(X:\mathbb{R})$. So $D(\mathcal{F})_{\rm loc} \cap L_{\infty;c}(X:\mathbb{R}) = D(\mathcal{E})_{\rm loc} \cap L_{\infty;c}(X:\mathbb{R})$.

Proposition 5.3. *Let $\mathcal{E}$ and $\mathcal{F}$ be local Dirichlet forms such that $\mathcal{E} \leq \mathcal{F}$. Assume $\mathbf{1} \in D(\mathcal{F})_{\rm loc}$, the space $D(\mathcal{F})$ is an ideal of $D(\mathcal{E})$ and $D(\mathcal{F}) \cap L_{\infty,c}(X)$ is dense in L_1. Then $d^{(\mathcal{F})}(A\,;B) \leq d^{(\mathcal{E})}(A\,;B)$ for all measurable $A, B \subset X$.*

Proof. It follows from the definition of the distance and Lemma 4.2 that it suffices to prove that $|||\widehat{\mathcal{I}}^{(\mathcal{E})}_\psi||| \leq |||\widehat{\mathcal{I}}^{(\mathcal{F})}_\psi|||$ for all positive $\psi \in D(\mathcal{F})_{\rm loc} \cap L_\infty(X:\mathbb{R})$.

First, one has again $\widehat{\mathcal{I}}^{(\mathcal{E})}_\psi(\varphi) \leq \widehat{\mathcal{I}}^{(\mathcal{F})}_\psi(\varphi)$ for all $\psi \in D(\mathcal{F})_{\rm loc} \cap L_\infty(X:\mathbb{R})$ and $\varphi \in D(\mathcal{F}) \cap L_{\infty,c}(X:\mathbb{R})$ with $\varphi \geq 0$.

Fix a positive $\psi \in D(\mathcal{F})_{\rm loc} \cap L_\infty(X:\mathbb{R})$. If $|||\widehat{\mathcal{I}}^{(\mathcal{F})}_\psi||| = \infty$ there is nothing to prove. So we may assume $|||\widehat{\mathcal{I}}^{(\mathcal{F})}_\psi||| < \infty$.

Since $\mathbf{1} \in D(\mathcal{F})_{\rm loc}$ it follows that $\mathbf{1} \in D(\mathcal{E})_{\rm loc}$. It then follows from Lemma 4.2 that

$$\widehat{\mathcal{I}}^{(\mathcal{E})}_\psi(\varphi) = \widehat{\mathcal{I}}^{(\mathcal{E})}_{\mathbf{1}+\psi}(\varphi) = 4\,\widehat{\mathcal{I}}^{(\mathcal{E})}_{(\mathbf{1}+\psi)^{1/2}}((\mathbf{1}+\psi)\varphi)$$

for all $\varphi \in D(\mathcal{E}) \cap L_{\infty;c}(X:\mathbb{R})$ by the Leibniz rule. But $(\mathbf{1}+\psi)\varphi \in D(\mathcal{F})_{\rm loc} \cap L_{\infty;c}(X:\mathbb{R})$ by the ideal property. Therefore

$$\widehat{\mathcal{I}}^{(\mathcal{E})}_\psi(\varphi) \leq 4\,\widehat{\mathcal{I}}^{(\mathcal{F})}_{(\mathbf{1}+\psi)^{1/2}}((\mathbf{1}+\psi)\varphi)$$

for all positive $\varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X:\mathbb{R})$. Hence

$$|||\widehat{\mathcal{I}}^{(\mathcal{E})}_\psi||| \leq 4\,(1+\|\psi\|_\infty)\,|||\widehat{\mathcal{I}}^{(\mathcal{F})}_{(\mathbf{1}+\psi)^{1/2}}||| \leq 4\,(1+\|\psi\|_\infty)\,|||\widehat{\mathcal{I}}^{(\mathcal{F})}_\psi|||$$

where the last inequality follows from Lemma 4.2.II. In particular $|||\widehat{\mathcal{I}}^{(\mathcal{E})}_\psi||| < \infty$.

Finally, since $|||\widehat{\mathcal{I}}_\psi^{(\mathcal{E})}||| < \infty$ one has

$$|\widehat{\mathcal{I}}_\psi^{(\mathcal{E})}(\varphi)| \leq \widehat{\mathcal{I}}_\psi^{(\mathcal{E})}(|\varphi|) \leq |||\widehat{\mathcal{I}}_\psi^{(\mathcal{E})}|||\, \|\varphi\|_1$$

for all $\varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X:\mathbb{R})$. But then there is a $\Gamma \in L_\infty$ such that $\|\Gamma\|_\infty = |||\widehat{\mathcal{I}}_\psi^{(\mathcal{E})}|||$ and

$$\widehat{\mathcal{I}}_\psi^{(\mathcal{E})}(\varphi) = \int \Gamma\, \varphi$$

for all $\varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X:\mathbb{R})$. Therefore

$$\Big| \int \Gamma\, \varphi \Big| = |\widehat{\mathcal{I}}_\psi^{(\mathcal{E})}(\varphi)| \leq |\widehat{\mathcal{I}}_\psi^{(\mathcal{E})}(|\varphi|)| \leq |\widehat{\mathcal{I}}_\psi^{(\mathcal{F})}(|\varphi|)| \leq |||\widehat{\mathcal{I}}_\psi^{(\mathcal{F})}|||\, \|\varphi\|_1$$

for all $\varphi \in D(\mathcal{F}) \cap L_{\infty,c}(X:\mathbb{R})$. Since the latter space is dense in $L_1(X:\mathbb{R})$ one deduces that $\|\Gamma\|_\infty \leq |||\widehat{\mathcal{I}}_\psi^{(\mathcal{F})}|||$. Hence $|||\widehat{\mathcal{I}}_\psi^{(\mathcal{E})}||| \leq |||\widehat{\mathcal{I}}_\psi^{(\mathcal{F})}|||$. The required monotonicity of the distances is immediate. $\square$

The proposition allows comparison of elliptic operators with different boundary conditions.

Example 5.4. Let X be an open subset of $\mathbb{R}^d$. Define the forms of the Neumann and Dirichlet Laplacians on $L_2(X)$ by $h_N(\varphi) = \|\nabla\varphi\|_2^2$ with $D(h_N) = W^{1,2}(X)$ and $h_D(\varphi) = \|\nabla\varphi\|_2^2$ with $D(h_D) = W_0^{1,2}(X)$. Then $h_N \leq h_D$, $\mathbf{1} \in D(h_D)_{\rm loc}$, $D(h_D)$ is an ideal of $D(h_N)$ and $D(h_D)\cap L_{\infty,c}(X)$ is dense in $L_1(X)$. Therefore one deduces from Proposition 5.3 that $d^{(h_D)}(A\,;B) \leq d^{(h_N)}(A\,;B)$ for all measurable $A, B \subseteq X$. Note that Proposition 5.1 does not apply to this example since $W_0^{1,2}(X)\cap L_{\infty,c}(X)$ is not a core for $D(h_N)$.

The argument used in Example 5.2 allows one to identify the distances associated with h_N and h_D with the geodesic distance in the Euclidean metric. In particular the distance is independent of the boundary conditions.

Example 5.5. Let X be an open subset of $\mathbb{R}^d$ and A, B non-empty open subsets of X. If X is disconnected then $d^{(h_N)}(A\,;B) = \infty = d^{(h_D)}(A\,;B)$ whenever A and B are in separate components by Theorem 1.3. Hence we may assume X is connected.

Next if $\psi, \varphi \in W^{1,2}(X) \cap L_\infty(X:\mathbb{R})$ then by direct calculation

$$\mathcal{I}_\psi^{(h_N)}(\varphi) = (\nabla(\varphi\psi), \nabla\psi) - 2^{-1}(\nabla\varphi, \nabla(\psi^2)) = \int_X dx\, \varphi(x)\, |(\nabla\psi)(x)|^2 \quad .$$

Therefore if $\psi \in W^{1,2}(X)_{\rm loc} \cap L_\infty(X:\mathbb{R})$ and $|||\widehat{\mathcal{I}}_\psi^{(h_N)}||| \leq 1$ one finds as in Example 5.2 that $\psi \in W^{1,\infty}(X:\mathbb{R})$ and $\|\nabla\psi\|_\infty \leq 1$. Hence $d^{(h_N)}(A\,;B) = \inf_{x\in A, y\in B} d(x\,;y)$ where

$$d(x\,;y) = \sup\{|\psi(x) - \psi(y)| : W^{1,\infty}(X:\mathbb{R})\,,\ \|\nabla\psi\|_\infty \leq 1\} \quad .$$

But this is the geodesic distance, with the usual Euclidean metric, from x to y. A similar conclusion follows for $d^{(h_D)}$ by replacing $W^{1,2}$ by $W_0^{1,2}$ in the argument.

This replacement does not affect the identification with the geodesic distance. Therefore in both cases the set-theoretic distance between the sets is the geodesic distance in X equipped with the Euclidean Riemannian metric.

The definition of the distance in terms of the space $D(\mathcal{E})_{\rm loc}$ gives good off-diagonal bounds but it is somewhat complicated. One could ask whether it has any simpler characterization in terms of $D(\mathcal{E})$. One obvious choice is to set

$$d_1'(A\,;B) = \sup\{d_\psi(A\,;B) : \psi \in D_1(\mathcal{E})\} \quad ,$$

for all measurable $A, B \subseteq X$ with $\overline{A}, \overline{B}$ compact, where

$$D_1(\mathcal{E}) = \{\psi \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R}) : |||\widehat{\mathcal{I}}_\psi||| \leq 1\} \quad .$$

Then one sets

$$\begin{aligned} d_1^{(\mathcal{E})}(A\,;B) &= d_1(A\,;B) \\ &= \inf\{d_1'(A_0\,;B_0) : A_0 \subseteq A,\ B_0 \subseteq B \text{ and } \overline{A_0}, \overline{B_0} \text{ are compact}\} \end{aligned}$$

for all measurable $A, B \subseteq X$. Note that $d_1(A\,;B) = d_1'(A\,;B)$ if $\overline{A}, \overline{B}$ are compact. It follows that in general $d^{(\mathcal{E})}(A\,;B) \geq d_1^{(\mathcal{E})}(A\,;B)$ but one can have a strict inequality.

Example 5.6. Let $\mathcal{E}$ be the form associated with the second-order operator $-d^2/dx^2$ with Dirichlet boundary conditions on the interval $\langle 0, 1\rangle$. If $A = \langle 0, a\rangle$ and $B = \langle b, 1\rangle$ with $0 < a < b < 1$ then the boundary conditions ensure that $d_\psi(A_\varepsilon\,;B_\varepsilon) \leq 2\varepsilon$ for all $\psi \in D(\mathcal{E}) \cap L_\infty(\langle 0,1\rangle : \mathbb{R})$ and small $\varepsilon > 0$, where $A_\varepsilon = \langle \varepsilon, a\rangle$ and $B_\varepsilon = \langle b, 1-\varepsilon\rangle$. Therefore $d_1(A\,;B) = 0$. But if $\psi(x) = 0 \vee (x-a) \wedge (b-a)$ then $\psi \in D_0(\mathcal{E})$ and $d(A\,;B) \geq d_\psi(A\,;B) = b - a$.

The two distances are, however, often equal.

Proposition 5.7. *Let $\mathcal{E}$ be a local Dirichlet form on $L_2(X)$. Suppose that for all compact K and $\varepsilon > 0$ there exists a $\chi \in D(\mathcal{E}) \cap L_{\infty,c}(X:\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi|_K = 1$ and $|||\widehat{\mathcal{I}}_\chi||| < \varepsilon$. Then $d(A\,;B) = d_1(A\,;B)$ for all measurable $A, B \subseteq X$ with $\overline{A}, \overline{B}$ compact.*

Proof. We have to show that $d(A\,;B) \leq d_1(A\,;B)$. Let $M \in [0, \infty\rangle$ and suppose that $M \leq d(A\,;B)$. Let $\varepsilon \in \langle 0, 1]$. By Lemma 4.2.III there exists a $\psi \in D_0(\mathcal{E})$ such that $d_\psi(A\,;B) \geq M - \varepsilon$ and $\|\psi\|_\infty \leq M$. By assumption there exists a $\chi \in D(\mathcal{E}) \cap L_{\infty,c}(X:\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi|_{\overline{A \cup B}} = 1$ and $|||\widehat{\mathcal{I}}_\chi||| < \varepsilon^2$. Then $\chi\psi \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$ and it follows from Lemma 3.3.II that

$$|||\widehat{\mathcal{I}}_{\chi\psi}||| \leq (1+\delta)\,\|\chi\|_\infty^2\,|||\widehat{\mathcal{I}}_\psi||| + (1+\delta^{-1})\,\|\psi\|_\infty^2\,|||\widehat{\mathcal{I}}_\chi|||$$

for all $\delta > 0$. Choosing $\delta = \varepsilon$ gives

$$|||\widehat{\mathcal{I}}_{\chi\psi}||| \leq (1+\varepsilon) + (1+\varepsilon^{-1})M^2\varepsilon^2 \leq 1 + \varepsilon(1+2M^2) \quad .$$

But $d_{\chi\psi}(A\,;B) = d_\psi(A\,;B) \geq M - \varepsilon$. So $d_1(A\,;B) \geq (1+\varepsilon(1+2M^2))^{-1}(M-\varepsilon)$. Hence $d_1(A\,;B) \geq M$ and the proposition follows. □

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On R-boundedness of Unions of Sets of Operators

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Dedicated to Philippe Clément on the occasion of his retirement

Abstract. It is shown that the union of a sequence $\mathcal{T}_1, \mathcal{T}_2, \ldots$ of R-bounded sets of operators from X into Y with R-bounds $\tau_1, \tau_2, \ldots$, respectively, is R-bounded if X is a Banach space of cotype q, Y a Banach space of type p, and $\sum_{k=1}^{\infty} \tau_k^r < \infty$, where $r = pq/(q-p)$ if $q < \infty$ and $r = p$ if $q = \infty$. Here $1 \leq p \leq 2 \leq q \leq \infty$ and $p \neq q$. The power r is sharp. Each Banach space that contains an isomorphic copy of c_0 admits operators $T_1, T_2, \ldots$ such that $\|T_k\| = 1/k$, $k \in \mathbb{N}$, and $\{T_1, T_2, \ldots\}$ is not R-bounded. Further it is shown that the set of positive linear contractions in an infinite-dimensional L^p is R-bounded only if $p = 2$.

1. Introduction

During the past few years a theory of L^p-multipliers for operator valued functions has been developed by means of the notion of R-boundedness of sets of operators, see for example [1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 17, 18, 19]. This theory has been applied to study maximal regularity of certain abstract evolution equations. For instance, L. Weis has shown in [18] that maximal L^p-regularity of the abstract Cauchy problem

$$\begin{aligned} &u'(t) = Au(t) + f(t) \quad \text{for a.e. } t \geq 0, \\ &u(0) = 0, \end{aligned} \tag{1.1}$$

in a Banach space X is equivalent to R-boundedness of the operator set

$$\{\lambda^n (\lambda I - A)^{-n} : \lambda \in i\mathbb{R}\}$$

for some $n \in \mathbb{N} = \{1, 2, \ldots\}$, whenever A is the generator of a bounded analytic semigroup on X and X is a UMD-space. Similarly, Arendt and Bu have shown in

[1] that maximal L^p-regularity of the problem

$$\begin{aligned} & u''(t) + Au(t) = f(t) \quad \text{for a.e. } t \in [0, \pi] \\ & u(0) = u(\pi) = 0 \end{aligned} \tag{1.2}$$

comes down to R-boundedness of

$$\{k^2(k^2 I - A)^{-1} : k \in \mathbb{N}\}.$$

Whether or not these sets of operators are R-bounded depends on the space X and the operator A that are considered. In order to be able to establish R-boundedness in specific cases, it seems useful to have a rich theory on manipulations with R-bounded sets at one's disposal.

This paper concerns unions of R-bounded sets in the Banach spaces $X = L^q(\mu)$, where $(A, \mathcal{A}, \mu)$ is an arbitrary measure space, and $1 \le q < \infty$. More generally, Banach spaces X and Y are considered where X has cotype q and Y has type p. It will be shown in Section 3 that the union of a sequence $\mathcal{T}_1, \mathcal{T}_2, \ldots$ of R-bounded sets of operators in $\mathcal{L}(X, Y)$ with R_2-bounds $\tau_1, \tau_2, \ldots$, respectively, is R-bounded if $\sum_{k=1}^\infty \tau_k^r < \infty$, where $r = pq/(q-p)$ and $1 \le p \le 2 \le q \le \infty$, $p \ne q$. In particular, a sequence of operators $\{T_1, T_2, \ldots\}$ on $L^p(\mu)$, $1 \le p < \infty$, is R-bounded whenever $\sum_{k=1}^\infty \|T_k\|^2 < \infty$. In Section 4 it is shown that the power r in the union theorem of Section 3 is sharp for $X = \ell^q(\mathbb{N})$ and $Y = \ell^p(\mathbb{N})$. It is also shown that Banach spaces that contain a copy of c_0 admit operators $T_1, T_2, \ldots$ such that $\|T_k\| = 1/k$, $k \in \mathbb{N}$, and $\{T_1, T_2, \ldots\}$ is not R-bounded. The ideas of Section 4 yield a characterization of $L^2(\mu)$ among $L^p(\mu)$, $1 \le p \le \infty$, by means of R-boundedness of positive contractions and isometries. This is discussed in Section 6. Section 7, finally, presents two examples related to resolvent families.

2. Preparations

Let X and Y be a real vector spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and let $\mathcal{L}(X, Y)$ denote the space of bounded linear operators that map X into Y. Let $1 \le r < \infty$. A subset $\mathcal{T}$ of $\mathcal{L}(X, Y)$ is called *R_r-bounded* if there exists a number $\tau_r \ge 0$ such that

$$\left(2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left\| \sum_{k=1}^n \varepsilon_k T_k x_k \right\|_Y^r \right)^{1/r} \le \tau_r \left(2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|_X^r \right)^{1/r}$$

for every $n \in \mathbb{N}$, every $x_1, \ldots, x_n \in X$ and every $T_1, \ldots, T_n \in \mathcal{T}$. The number τ_r is called an R_r-bound of $\mathcal{T}$. We always assume that the spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces. Kahane's inequality (see [15, Theorem 1.e.13, p. 74]) says that for every $p, q \in [1, \infty)$ there exists a constant $K_{p,q}(X) \ge 0$ such that

$$\left(2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|_X^p \right)^{1/p} \le K_{p,q}(X) \left(2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|_X^q \right)^{1/q}$$

for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in X$. Thus a set $\mathcal{T} \subset \mathcal{L}(X,Y)$ is R_r-bounded for every $r \in [1,\infty)$ as soon as it is R_r-bounded for some $r \in [1,\infty)$ and therefore it is then simply called *R-bounded*.

We will focus on Banach spaces X that have *cotype* $q \in [2,\infty]$ and Banach spaces Y that have *type* $p \in [1,2]$, that is, there exist constants $M_X, M_Y \geq 0$ such that for every $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, and $y_1, \ldots, y_n \in Y$ the inequalities

$$\left(\sum_{k=1}^{n} \|x_k\|_X^q \right)^{1/q} \leq M_X \left(2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|_X^2 \right)^{1/2}$$

and

$$\left(2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left\| \sum_{k=1}^{n} \varepsilon_k y_k \right\|_Y^2 \right)^{1/2} \leq M_Y \left(\sum_{k=1}^{n} \|y_k\|_Y^p \right)^{1/p}$$

hold, respectively. The expression $(\sum_{k=1}^{n} \|x_k\|_X^q)^{1/q}$ should be replaced by

$$\max_{1 \leq k \leq n} \|x_k\|_X$$

if $q = \infty$. The constants M_X and M_Y are called a *cotype q constant* of X and a *type p constant* of Y, respectively. Each Banach space is of type 1 and cotype ∞ with both type 1 and cotype ∞ constant equal to 1. If a Banach space is of type p then it is of type r for any $r \in [1,p]$ and if it is of cotype q then it is of cotype r for any $r \in [q,\infty]$. For an arbitrary measure space $(A, \mathcal{A}, \mu)$ and $1 \leq p < \infty$, the Banach space $L^p(\mu)$ is of type $p \wedge 2 = \min\{p,2\}$ and of cotype $q \vee 2 = \max\{q,2\}$. See [15, p. 72–73] for proofs and more details.

Every R-bounded subset of $\mathcal{L}(X,Y)$ is bounded. The converse is true if and only if X is of cotype 2 and Y is of type 2 (see [1, Proposition 1.13]).

In order to abbreviate notations, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and independent identically distributed (i.i.d) random variables $\varepsilon_1, \varepsilon_2, \ldots$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}(\varepsilon_1 = -1) = \mathbb{P}(\varepsilon_1 = 1) = 1/2$. Expectation is denoted by $\mathbb{E}$. Then, for instance,

$$\mathbb{E} \left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|_X^r = 2^{-n} \sum_{\delta \in \{-1,1\}^n} \left\| \sum_{k=1}^{n} \delta_k x_k \right\|_X^r = \left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|_{L^r(\mathbb{P},X)}^r .$$

The next proposition collects some preliminary results which are needed in the sequel (see also [5, 7, 12, 19]).

Proposition 2.1. *Let X, Y, and Z be Banach spaces, let $1 \leq r < \infty$, and let $\mathcal{S}, \mathcal{T} \subset \mathcal{L}(X,Y)$ and $\mathcal{U} \subset \mathcal{L}(Y,Z)$ be R_r-bounded by σ, τ, and u, respectively.*

1. *$\{T\}$ is R_r-bounded by $\|T\|$, for every $T \in \mathcal{L}(X,Y)$.*
2. *If $\mathcal{S} \subset \mathcal{T}$, then $\mathcal{S}$ is R_r-bounded by τ.*
3. *$\mathcal{U}\mathcal{S} = \{US : U \in \mathcal{U}, S \in \mathcal{S}\}$ is R_r-bounded by $u\sigma$.*
4. *$\mathcal{T} \cup \{0\}$ is R_r-bounded by τ.*
5. *$\mathcal{S} + \mathcal{T} = \{S + T : S \in \mathcal{S}, T \in \mathcal{T}\}$ is R_r-bounded by $\sigma + \tau$.*

6. *$\mathcal{S} \cup \mathcal{T}$ is R_r-bounded by $\sigma + \tau$.*
7. *If Λ is a directed partially ordered set and $(\mathcal{T}_\lambda)_{\lambda \in \Lambda}$ is an increasing family of subsets of $\mathcal{L}(X, Y)$ such that $\mathcal{T}_\lambda$ is R_r-bounded by τ for every $\lambda \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} \mathcal{T}_\lambda$ is R_r-bounded by τ.*

It follows from 2 and 6 of Proposition 2.1 that the notion of R-boundedness defines a bornology in $\mathcal{L}(X, Y)$ and thus the word 'boundedness' is justified (see [11]).

The R_r-bound for the union $\mathcal{S} \cup \mathcal{T}$ given in 6 of Proposition 2.1 may seem rather large. It turns out, however, that this bound is the best bound that holds in general. In particular, a countable set $\{T_1, T_2, \ldots\}$ need not be R-bounded if the norms $\|T_1\|, \|T_2\|, \ldots$ are not summable (see Theorem 5.1). On the other hand, in a Hilbert space a union of R-bounded sets is R-bounded if their R-bounds are bounded. It may be expected that for $X = Y = L^p$ the conditions providing R-boundedness of a union of R-bounded sets is somewhere between boundedness and summability of the R-bounds of the components. That is, more towards boundedness if p is close to 2 and more towards summability if p is large. Theorem 3.1 shows that this expectation is true.

3. R-boundedness of unions

The proof of the next theorem uses the following observation on type and cotype of Bochner spaces. If $(A, \mathcal{A}, \mu)$ is a measure space and X a Banach space of type p, then $L^s(\mu, X)$ is of type p for each $p \le s < \infty$. If X is of cotype q, then so is $L^s(\mu, X)$ for each $1 \le s \le q$ (see [16, Lemme 1.1, Lemme 2.1]). More specifically, if X is of type p with type p constant M, then $L^2(\mu, X)$ also has type p constant M, and if X is of cotype q with cotype q constant M, then also $L^2(\mu, X)$ has cotype q constant M. These facts can easily be proved by means of the inequality

$$\left(\sum_{k=1}^{n} \left(\int |f_k(a)| \, \mathrm{d}\mu(a) \right)^p \right)^{1/p} \le \int \left(\sum_{k=1}^{n} |f_k(a)|^p \right)^{1/p} \mathrm{d}\mu(a),$$

which holds for $p \ge 1$, and the reversed inequality, which holds for $0 < p \le 1$, $n \in \mathbb{N}$, $f_1, \ldots, f_n \in L^1(\mu)$.

Theorem 3.1. *Let $1 \le p \le 2 \le q \le \infty$ be such that $p \ne q$. Let $(X, \|\cdot\|_X)$ be a Banach space of cotype q and $(Y, \|\cdot\|_Y)$ a Banach space of type p. Let $\mathcal{T}_1, \mathcal{T}_2, \ldots$ be subsets of $\mathcal{L}(X, Y)$ such that for each $k \in \mathbb{N}$ the set $\mathcal{T}_k$ is R_2-bounded by a number $\tau_k > 0$. Let $\mathcal{T} := \bigcup_{k=1}^\infty \mathcal{T}_k$ and denote*

$$r := r_{p,q} := \frac{pq}{q-p} \text{ if } q < \infty \quad \text{and} \quad r_{p,q} := p \text{ if } q = \infty.$$

If $R := (\sum_{k=1}^\infty \tau_k^r)^{1/r}$ is finite, then $\mathcal{T}$ is R_2-bounded by $M_Y M_X R$, where M_X is a cotype q constant of X and M_Y a type p constant of Y.

Proof. Let $n \in \mathbb{N}$, $T_1, \dots, T_n \in \mathcal{T}$ and let $x_1, \dots, x_n \in X$. Choose a partition $I_1, \dots I_K$ of $\{1, \dots, n\}$ satisfying

$$T_i \in \mathcal{T}_k \text{ for all } i \in I_k,\ k = 1, \dots, K.$$

Here K is a finite number and some of the sets I_k may be empty. Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a probability space and let $\varepsilon'_{k,i}$, $i \in I_k$, $k = 1, \dots, K$, be i.i.d. random variables on $(\Omega', \mathcal{F}', \mathbb{P}')$ with $\mathbb{P}'(\varepsilon'_{k,i} = -1) = \mathbb{P}'(\varepsilon'_{k,i} = 1) = 1/2$. Then also the products $\varepsilon_k \varepsilon'_{k,i}$, $i \in I_k$, $k = 1, \dots, K$, (in the usual way extended to $\Omega \times \Omega'$) are i.i.d. random variables with uniform distribution on $\{-1, 1\}$. Denote expectation with respect to $\mathbb{P}'$ by $\mathbb{E}'$.

Let us start the estimations. In the next sequence of inequalities we use consecutively type p of $L^2(\mathbb{P}', Y)$, R-boundedness, Hölder's inequality, and cotype q of $L^2(\mathbb{P}', X)$. We obtain for $q < \infty$,

$$\begin{aligned}
\left(\mathbb{E}\left\|\sum_{k=1}^{n} \varepsilon_k T_k x_k\right\|_Y^2\right)^{1/2} &= \left(\mathbb{E}\,\mathbb{E}'\left\|\sum_{k=1}^{K}\sum_{i\in I_k} \varepsilon_k \varepsilon'_{k,i} T_i x_i\right\|_Y^2\right)^{1/2} \\
&= \left(\mathbb{E}\left\|\sum_{k=1}^{K} \varepsilon_k \sum_{i\in I_k} \varepsilon'_{k,i} T_i x_i\right\|_{L^2(\mathbb{P}',Y)}^2\right)^{1/2} \\
&\le M_Y \left(\sum_{k=1}^{K} \left\|\sum_{i\in I_k} \varepsilon'_{k,i} T_i x_i\right\|_{L^2(\mathbb{P}',Y)}^p\right)^{1/p} \\
&\le M_Y \left(\sum_{k=1}^{K} \tau_k^p \left\|\sum_{i\in I_k} \varepsilon'_{k,i} x_i\right\|_{L^2(\mathbb{P}',X)}^p\right)^{1/p} \\
&\le M_Y R \left(\sum_{k=1}^{K} \left\|\sum_{i\in I_k} \varepsilon'_{k,i} x_i\right\|_{L^2(\mathbb{P}',X)}^q\right)^{1/q} \\
&\le M_Y R M_X \left(\mathbb{E}\left\|\sum_{k=1}^{K} \varepsilon_k \sum_{i\in I_k} \varepsilon'_{k,i} x_i\right\|_{L^2(\mathbb{P}',X)}^2\right)^{1/2} \\
&= M_Y R M_X \left(\mathbb{E}\,\mathbb{E}'\left\|\sum_{k=1}^{K}\sum_{i\in I_k} \varepsilon_k \varepsilon'_{k,i} x_i\right\|_X^2\right)^{1/2} \\
&= M_Y R M_X \left(\mathbb{E}\left\|\sum_{k=1}^{n} \varepsilon_k x_k\right\|_X^2\right)^{1/2}.
\end{aligned}$$

If $q = \infty$, we only have to replace

$$\left(\sum_{k=1}^{K} \left\| \sum_{i \in I_k} \varepsilon'_{k,i} x_i \right\|_{L^2(\mathbb{P}',X)}^{q} \right)^{1/q}$$

by $\max_{1 \leq k \leq K} \| \sum_{i \in I_k} \varepsilon'_{k,i} x_i \|_{L^2(\mathbb{P}',X)}$ and recall that any Banach space is of cotype ∞. $\square$

The case $p = q = 2$, which is not treated in the previous theorem follows directly from the definitions. If X is of cotype 2 and Y is of type 2, then every bounded subset of $\mathcal{L}(X, Y)$ is R-bounded. Moreover, if M_X is a cotype 2 constant of X and M_Y is a type 2 constant of Y and if $\mathcal{T} \subset \mathcal{L}(X, Y)$ and $\tau \in \mathbb{R}$ are such that $\|T\| \leq \tau$ for all $T \in \mathcal{T}$, then $M_X M_Y \tau$ is an R_2-bound of $\mathcal{T}$. Further notice that the cases $p > 2$ and $q < 2$ are not of interest, since the only Banach space with type $p > 2$ or cotype $q < 2$ is $\{0\}$ (see[15, p. 73]).

Corollary 3.2. *Let X be a Banach space of type 2 or cotype 2. Let $\mathcal{T}_1, \mathcal{T}_2, \ldots$ be subsets of $\mathcal{L}(X)$ such that for each k the set $\mathcal{T}_k$ is R-bounded by $\tau_k > 0$. If $\sum_{k=1}^{\infty} \tau_k^2 < \infty$, then $\bigcup_{k=1}^{\infty} \mathcal{T}_k$ is R-bounded.*

Corollary 3.3. *Let $1 \leq p \leq 2 \leq q \leq \infty$ be such that $p \neq q$. Let X be a Banach space of cotype q and let Y be a Banach space of type p. Let $T_1, T_2, \ldots \in \mathcal{L}(X, Y)$. If $\sum_{k=1}^{\infty} \|T_k\|^{r_{p,q}} < \infty$, then $\{T_1, T_2, \ldots\}$ is R-bounded and an R_2-bound of this set is $M_Y M_X (\sum_{k=1}^{\infty} \|T_k\|^{r_{p,q}})^{1/r_{p,q}}$, where $r_{p,q}$, M_Y, and M_X are as in Theorem 3.1.*

Example 3.4. (Cesaro means) Let $1 \leq p \leq 2 \leq q \leq \infty$, $p \neq q$, and either $p > 1$ or $q < \infty$, and let X be a Banach space of type p and cotype q. Let M_1 be a type p constant and M_2 be a cotype q constant. Let $T \in \mathcal{L}(X)$ be such that $\|T\| \leq 1$ and $1 \in \rho(T)$. Let

$$S_m := \frac{1}{m} \sum_{k=1}^{m} T^k, \quad m = 1, 2, \ldots.$$

Then the set $\{S_1, S_2, \ldots\}$ is R-bounded. More specifically, for each $n \in \mathbb{N}$ the set $\{S_1, \ldots, S_n\}$ is R_2-bounded by

$$C \sup_{1 \leq m \leq n} \|T(I - T^m)(I - T)^{-1}\|,$$

where $C = M_1 M_2 (\sum_{m=1}^{\infty} (1/m)^{r_{p,q}})^{1/r_{p,q}}$. Indeed, it is clear that

$$\frac{1}{m}(T + \cdots + T^m) = \frac{1}{m} T(I - T^m)(I - T)^{-1}.$$

Corollary 3.3 yields that the set $\{S_1, \ldots, S_n\}$ is R_2-bounded by

$$C \sup_{1 \leq m \leq n} \|T(I - T^m)(I - T)^{-1}\|.$$

Since $p > 1$ or $q < \infty$, we have $r_{p,q} > 1$ and C is finite. The proof is completed by the observation that the sets $\{S_1, \ldots, S_n\}$ are increasing in n and that for all m

$$\|T(I - T^m)(I - T)^{-1}\| \leq \|T\|(\|I\| + \|T\|^m)\|(I - T)^{-1}\| \leq 2\|(I - T)^{-1}\|.$$

4. Sharpness of the powers

We will show next that the power $r_{p,q}$ in Theorem 3.1 is sharp. That means, if $1 \le p \le 2 \le q \le \infty$, $p \ne q$, and $s > r_{p,q}$, then there are Banach spaces X and Y such that X is of cotype q and Y of type p and there are sets $\mathcal{T}_1, \mathcal{T}_2, \ldots$ in $\mathcal{L}(X, Y)$ with R$_2$-bounds $\tau_1, \tau_2, \ldots$, respectively, such that $\sum_{k=1}^{\infty} \tau_k^s < \infty$ but $\bigcup_{k=1}^{\infty} \mathcal{T}_k$ is not R-bounded.

In fact we show more. If $1 \le p \le \infty$, $1 \le q \le \infty$, not $q \le 2 \le p$, and $s > r_{p,q}$ where

$$r_{p,q} := \frac{(p \wedge 2)(q \vee 2)}{(q \vee 2) - (p \wedge 2)} \text{ if } q < \infty \quad \text{and} \quad r_{p,q} := p \wedge 2 \text{ if } q = \infty,$$

then there exist $T_1, T_2, \ldots \in \mathcal{L}(\ell^q(\mathbb{N}), \ell^p(\mathbb{N}))$ such that $\sum_{k=1}^{\infty} \|T_k\|^s < \infty$ and $\{T_1, T_2, \ldots\}$ not R-bounded.

The analysis is based on Khintchine's inequality (see [14, Theorem 2.b.3, p. 66]), which says that for any $1 \le r < \infty$ there exist constants c_r and C_r in $(0, \infty)$ such that

$$c_r \left(\sum_{k=1}^{n} |\alpha_k|^2 \right)^{1/2} \le \left(\mathbb{E} \left| \sum_{k=1}^{n} \varepsilon_k \alpha_k \right|^r \right)^{1/r} \le C_r \left(\sum_{k=1}^{n} |\alpha_k|^2 \right)^{1/2}, \tag{4.1}$$

for all $n \in \mathbb{N}$ and all $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.

Lemma 4.1. *Let $(A, \mathcal{A}, \mu)$ be a measure space, let $1 \le p \le \infty$ and $1 \le q \le \infty$, let $1 \le r < \infty$, $n \in \mathbb{N}$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Suppose that there exist $e_1, \ldots, e_n \in L^p(\mu) \cap L^q(\mu)$ with*

$$e_k \ge 0, \ e_k \ne 0, \ e_k \wedge e_\ell = 0 \text{ for all } \ell \ne k \text{ and } k = 1, \ldots, n.$$

Let $T_1, \ldots, T_n \in \mathcal{L}(L^q(\mu), L^p(\mu))$ and let τ be an R_r-bound of $\{\alpha_1 T_1, \ldots, \alpha_n T_n\}$.

1. *If $T_k e_k = e_1$ and $\|e_k\|_q = 1$ for $k = 1, \ldots, n$, then*
$$\tau \ge c_r \|e_1\|_p n^{-1/q} \|(\alpha_1, \ldots, \alpha_n)\|_2$$
if $q < \infty$ and $\tau \ge c_r \|e_1\|_p \|(\alpha_1, \ldots, \alpha_n)\|_2$ if $q = \infty$.
2. *If $T_k e_1 = e_k$ and $\|e_k\|_p = 1$ for $k = 1, \ldots, n$, then*
$$\tau \ge C_r^{-1} \|e_1\|_q^{-1} n^{-1/2} \|(\alpha_1, \ldots, \alpha_n)\|_p.$$
3. *If $T_k e_k = e_k$ and $\|e_k\|_q = 1$ for $k = 1, \ldots, n$, then*
$$\tau \ge n^{-1/q} \|(\alpha_1 \|e_1\|_p, \ldots, \alpha_n \|e_n\|_p)\|_p$$
if $q < \infty$ and $\tau \ge \|(\alpha_1 \|e_1\|_p, \ldots, \alpha_n \|e_n\|_p)\|_p$ if $q = \infty$.

Here c_r and C_r are constants that satisfy (4.1).

Proof. For any $\beta_1, \ldots \beta_n \in \mathbb{R}$ and $s = p$ or $s = q$, disjointness of the e_k yields

$$\left(\mathbb{E} \left\| \sum_{k=1}^{n} \varepsilon_k \beta_k e_k \right\|_s^r \right)^{1/r} = \begin{cases} \left(\sum_{k=1}^{n} |\beta_k|^s \|e_k\|_s^s \right)^{1/s} & \text{if } s < \infty, \\ \max_{1 \le k \le n} |\beta_k| \|e_k\|_s & \text{if } s = \infty. \end{cases}$$

By (4.1), for any $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$ and $s = p$ or $s = q$,

$$\begin{aligned} \left(\mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k \gamma_k e_1 \right\|_s^r \right)^{1/r} &= \left(\mathbb{E} \left| \sum_{k=1}^n \varepsilon_k \gamma_k \right|^r \|e_1\|_s^r \right)^{1/r} \\ &\leq C_r \left(\sum_{k=1}^n \gamma_k^2 \right)^{1/2} \|e_1\|_s \end{aligned}$$

and $(\mathbb{E} \| \sum_{k=1}^n \varepsilon_k \gamma_k e_1 \|_s^r)^{1/r} \geq c_r (\sum_{k=1}^n \gamma_k^2)^{1/2} \|e_1\|_s$.

To show 1., choose $x_k = e_k$, $k = 1, \ldots, n$. Then $T_k x_k = e_1$, so

$$\left(\mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k \alpha_k T_k x_k \right\|_p^r \right)^{1/r} \geq c_r \left(\sum_{k=1}^n \alpha_k^2 \right)^{1/2} \|e_1\|_p.$$

Hence

$$c_r \|e_1\|_p \|(\alpha_1, \ldots, \alpha_n)\|_2 \leq \tau \left(\mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|_q^r \right)^{1/r} = \begin{cases} n^{1/q} & \text{if } q < \infty \\ 1 & \text{if } q = \infty \end{cases}$$

and the lower bound for τ follows.

Similarly, 2. and 3. are shown by choosing $x_k = e_1$ and $x_k = e_k$ for $k = 1, \ldots, n$, respectively. $\square$

Proposition 4.2. *Let $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ be such that not $p = q = 2$. Define for $k \in \mathbb{N}$ operators $S_k, T_k, U_k \in \mathcal{L}(\ell^q(\mathbb{N}), \ell^p(\mathbb{N}))$ by*

$$S_k x := x(k) \mathbb{1}_{\{1\}}, \quad T_k x := x(1) \mathbb{1}_{\{k\}}, \quad U_k := x(k) \mathbb{1}_{\{k\}}, \quad x \in \ell^q(\mathbb{N}).$$

1. *If $p \geq 2$, $q > 2$, and $s > r_{p,q}$, then $\{(1/k)^{1/s} S_k \colon k \in \mathbb{N}\}$ is not R-bounded.*
2. *If $p < 2$, $q \leq 2$, and $s > r_{p,q}$, then $\{(1/k)^{1/s} T_k \colon k \in \mathbb{N}\}$ is not R-bounded.*
3. *If $p \leq 2 \leq q$ and $s > r_{p,q}$, then $\{(1/k)^{1/s} U_k \colon k \in \mathbb{N}\}$ is not R-bounded.*

Proof. 1. For an $n \in \mathbb{N}$, choose $x_k = \mathbb{1}_{\{k\}}$, $k = 1, \ldots, n$. According to Lemma 4.1.1, an R_r-bound of $\{(1/k)^{1/s} S_k \colon k \in \mathbb{N}\}$ would be greater than $\frac{c_r}{n^{1/q}} (\sum_{k=1}^n k^{-2/s})^{1/2}$ if $q < \infty$ and greater than $c_r (\sum_{k=1}^n k^{-2/s})^{1/2}$ if $q = \infty$, for every $n \in \mathbb{N}$. As $(\sum_{k=1}^n k^{-2/s})^{1/2} \geq n^{1/2 - 1/s}$, and $1/2 - 1/s - 1/q > 0$, this is impossible.

2. For $n \in \mathbb{N}$, choose $x_k = \mathbb{1}_{\{1\}}$, $k = 1, \ldots, n$. According to Lemma 4.1.2, an R_r-bound of $\{(1/k)^{1/s} T_k \colon k \in \mathbb{N}\}$ would be at least $C_r^{-1} n^{-1/2} (\sum_{k=1}^n k^{-p/s})^{1/p}$, which is greater than $C_r^{-1} n^{-1/2 + 1/p - 1/s}$. This is impossible, as $-1/2 + 1/p - 1/s > 0$.

3. For $n \in \mathbb{N}$, choose $x_k = \mathbb{1}_{\{k\}}$, $k = 1, \ldots, n$. Lemma 4.1.3 yields that an R_r-bound of $\{(1/k)^{1/s} U_k \colon k \in \mathbb{N}\}$ would majorize $n^{-1/q} (\sum_{k=1}^n k^{-p/s})^{1/p}$ if $q < \infty$ and $(\sum_{k=1}^n k^{-p/s})^{1/p}$ if $q = \infty$, for all n. As $(\sum_{k=1}^n k^{-p/s})^{1/p} \geq n^{1/p - 1/s}$ and $-1/q + 1/p - 1/s > 0$, such an R_r-bound cannot exist. $\square$

It follows from the previous proposition that the power $r_{p,q}$ is sharp in Theorem 3.1 for $X = \ell^q(\mathbb{N})$ and $Y = \ell^p(\mathbb{N})$. Indeed, observe first that the operators S_k, T_k, and U_k are contractive from X into Y. In the case $p \geq 2$ and $q > 2$, we can choose for each $s > r_{p,q}$ a $t \in (r_{p,q}, s)$ and then $(1/k)^{1/t} S_k$ are operators in $\mathcal{L}(\ell^q, \ell^p)$ such that $\sum_{k=1}^{\infty} \|(1/k)^{1/t} S_k\|^s < \infty$ and $\{(1/k)^{1/t} S_k \colon k \in \mathbb{N}\}$ is not R-bounded, by Proposition 4.2.1. In the case $p < 2$ and $q \leq 2$ we consider $(1/k)^{1/t} T_k$, $k \in \mathbb{N}$, and use Proposition 4.2.2. If $p \leq 2 \leq q$ we use $(1/k)^{1/t} U_k$ and Proposition 4.2.3. Observe that in the remaining case $q \leq 2 \leq p$ the space $X = \ell^q(\mathbb{N})$ has cotype 2 and $Y = \ell^p(\mathbb{N})$ has type 2, so that every bounded subset of $\mathcal{L}(X, Y)$ is R-bounded.

5. Banach spaces containing $\mathbf{c_0}$

Theorem 3.1 might raise questions about converse implications. For instance, for which Banach spaces X is the following statement true: if $T_1, T_2, \ldots$ are bounded linear operators on X and $(\|T_k\|_k)_k \in \ell^r(\mathbb{N})$ for some $r > 1$, then $\{T_k \colon k \in \mathbb{N}\}$ is R-bounded. Corollary 3.3 yields that this property is true if X is of type $p > 1$ or of cotype $q < \infty$. We will show next that the above property does not hold for Banach spaces containing an isomorphic copy of c_0.

Theorem 5.1. *Let X be a Banach space that contains an isomorphic copy of c_0. Then there are $T_1, T_2, \ldots \in \mathcal{L}(X)$ such that $\|T_k\| = 1/k$ and such that $\{T_1, T_2, \ldots\}$ is not R-bounded. More specifically, one can arrange that there exists a constant $C > 0$ such that the R-bound of $\{T_1, \ldots, T_n\}$ is $\geq C \sum_{k=1}^n 1/k$ for every n.*

Proof. For clarity of the argument we first consider the case that $X = c_0$. Define for $n \geq 1$ and $0 \leq m \leq n$ the blocks of indices

$$I^n_{m,\ell} := \{i \in \mathbb{N} \colon\ 2^n + (\ell - 1)2^{n-m} \leq i < 2^n + \ell 2^{n-m}\}, \quad \ell = 1, \ldots, 2^m,$$

and the elements

$$y^n_m := \sum_{\ell=1}^{2^m} (-1)^\ell \mathbb{1}_{I^n_{m,\ell}}.$$

Define for $k = 1, 2, \ldots$ the operators

$$T_k x := \frac{1}{k} \sum_{n=0}^{\infty} x(2^n + k) y^n_k, \quad x \in c_0.$$

Since the elements y^n_k, $n \in \mathbb{N}$, have disjoint supports, we obtain $T_k \in \mathcal{L}(c_0)$. Further, for each $n \in \mathbb{N}$, $T_k \mathbb{1}_{\{2^n+k\}} = (1/k) y^n_k$, and $\|T_k\| = 1/k$, $k = 1, 2, \ldots$.

We claim that $\{T_1, T_2, \ldots\}$ is not R-bounded. Indeed, let $n \in \mathbb{N}$ and let $x_k = \mathbb{1}_{\{2^n+k\}}$, $k = 1, \ldots, n$. As the x_k are disjointly supported and $\|x_k\|_\infty = 1$, we have

$$2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|_\infty = 1.$$

On the other hand, if $\varepsilon \in \{-1,1\}^n$, then we can choose ℓ_1 such that $\varepsilon_1 y_1^n$ equals 1 on I^n_{1,ℓ_1}, and we can choose inductively ℓ_m such that $\varepsilon_m y_m^n$ equals 1 on I^n_{m,ℓ_m} and $I^n_{m,\ell_m} \subset I^n_{m-1,\ell_{m-1}}$, for $m = 2,\dots,n$. With $i \in I^n_{n,\ell_n}$ we find that

$$\left\| \sum_{m=1}^n \varepsilon_m (1/m) y_m \right\|_\infty \geq \sum_{m=1}^n \varepsilon_m (1/m) y_m(i) = \sum_{m=1}^n (1/m).$$

Thus,

$$2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \left\| \sum_{k=1}^n \varepsilon_k T_k x_k \right\|_\infty \geq \sum_{k=1}^n 1/k.$$

It follows that $\{T_1, T_2, \dots\}$ is not R-bounded. Actually we have shown that the smallest R_1-bound of $\{T_1,\dots,T_n\}$ equals $\sum_{k=1}^n 1/k = \sum_{k=1}^n \|T_k\|$.

Next, let X be a Banach space such that there exists a linear injection $i\colon c_0 \to X$ and constants $c, C > 0$ such that

$$c\|i(x)\|_X \leq \|x\|_\infty \leq C\|i(x)\|_X, \quad x \in c_0.$$

For $n, k \in \mathbb{N}$ the map $x \mapsto (i^{-1}(x))(2^n + k)$ is a bounded linear function on $i(c_0)$ with norm $\leq C$ and according to Hahn-Banach's theorem it can be extended to a bounded linear functional $\varphi_{n,k}$ on X with $\|\varphi_{n,k}\| \leq C$. Now define the operators

$$T_k x := (1/k) i \left(\sum_{n=0}^\infty \varphi_{n,k}(x) y_k^n \right), \quad x \in X,\ k = 1,2,\dots.$$

Then $T_k \in \mathcal{L}(X)$ and $\|T_k\| \leq k^{-1} c^{-1} C$ for all k. Further, $T_k(i(\mathbb{1}_{\{2^n+k\}})) = (1/k) i(y_k^n)$ for all $k, n \in \mathbb{N}$. Rescale T_k to $S_k := (1/k) T_k / \|T_k\|$, so that $\|S_k\| = 1/k$ for all k. If we fix $n \in \mathbb{N}$ and let

$$x_k := i(\mathbb{1}_{\{2^n+k\}}), \quad k = 1,\dots,n,$$

then

$$\left\| \sum_{k=1}^n \varepsilon_k S_k x_k \right\|_X \geq c^{-1} \sum_{k=1}^n k^{-2} \|T_k\|^{-1} \geq c C^{-2} \sum_{k=1}^n 1/k.$$

The assertions now follow easily. □

6. Characterization of $\mathbf{L^2}$

Lemma 4.1 leads to a characterization of L^2 among L^p be means of R-boundedness of positive linear contractions and isometries. By a linear *contraction* on a Banach space $(X, \|\cdot\|)$ we mean a linear map $T \in \mathcal{L}(X)$ with $\|T\| \leq 1$. The map $T \in \mathcal{L}(X)$ is an *isometry* if $\|Tx\| = \|x\|$ for all $x \in X$. If X is a Banach lattice, then we call a linear map $T : X \to X$ *positive* if $Tx \geq 0$ for all $x \geq 0$. If $(\Omega, \mathcal{F}, \mu)$ is a measure space, $1 \leq p \leq \infty$, and $\Omega_1 \in \mathcal{F}$, then $L^p(\mu)$ decomposes into $L^p(\mu) = L^p(\mu_1) \oplus L^p(\mu_2)$, where $\Omega_2 = \Omega \setminus \Omega_1$, $\mathcal{F}_i = \{A \cap \Omega_i\colon A \in \mathcal{F}\}$, $\mu_i(A) = \mu(A)$, $A \in \mathcal{F}_i$, $i = 1,2$, and the norm of $L^p(\mu_1) \oplus L^p(\mu_2)$ is given by $(\|f|_{\Omega_1}\|^p + \|f|_{\Omega_2}\|^p)^{1/p}$ if $p < \infty$ and $\|f|_{\Omega_1}\|_\infty \vee \|f|_{\Omega_2}\|_\infty$ if $p = \infty$. We further recall that an *atom* in a σ-finite measure space $(\Omega, \mathcal{F}, \mu)$ is a set $A \in \mathcal{F}$ with $\mu(A) > 0$ such that for every $E \in \mathcal{F}$ with $\mu(E \setminus A) = 0$ either $\mu(E) = \mu(A)$ or $\mu(E) = 0$.

Theorem 6.1. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, let $1 \leq p \leq \infty$, and let $X = L^p(\mu)$. Assume that X is infinite-dimensional.*

1. *The set of all positive linear contractions on X is R-bounded if and only if $p = 2$.*
2. *If X is separable, then the set of all positive linear surjective isometries on X is R-bounded if and only if $p = 2$.*

Proof. 1. Since in a Hilbert space every uniformly bounded set of operators is R-bounded, the 'if' part is clear. Suppose that $p \neq 2$. Let $n \in \mathbb{N}$ be arbitrary. Since $L^p(\mu)$ is infinite-dimensional, there exist mutually disjoint sets $A_1, \ldots, A_n \in \mathcal{F}$ with $\mu(A_i) > 0$ for $i = 1, \ldots, n$. Define $\alpha_i := \mu(A_i)^{-1/p}$ if $p < \infty$ and $\alpha_i := 1$ if $p = \infty$, for $i = 1, \ldots, n$. Further define

$$e_i := \alpha_i \mathbb{1}_{A_i} \text{ and } \varphi_i(f) := \alpha_i^{-1} \mu(A_i)^{-1} \int f \mathbb{1}_{A_i} \, \mathrm{d}\mu, \; f \in X,$$

$i = 1, \ldots, n$, and

$$Tf := \varphi_1(f) e_n + \varphi_2(f) e_1 + \varphi_3(f) e_2 + \cdots + \varphi_n(f) e_{n-1},$$

$f \in X$. It is easily checked that T is a positive linear contraction from X into X. Further, $Te_i = e_{i-1}$ for $i = 2, \ldots, n$ and $Te_1 = e_n$. Let $T_i := T^{i-1}$, $i = 1, \ldots, n$. Then Lemma 4.1 yields that any R_r-bound of $\{T_1, \ldots, T_n\}$ is at least $c_r n^{1/2-1/p} \vee C_r^{-1} n^{1/p-1/2}$ if $p < \infty$ and at least $c_r n^{1/2}$ if $p = \infty$. It follows that the set of positive linear contractions is not R-bounded.

2. As before, the 'if' part is well known. Suppose that $p \neq 2$. Since $L^p(\mu)$ is infinite-dimensional, there are mutually disjoint sets $A_1, A_2, \ldots \in \mathcal{F}$ with $0 < \mu(A_i) < \infty$ for all i. Then $\Omega_1 := \bigcup_{i=1}^{\infty} A_i$ with $\mathcal{F}_1 = \{A \cap \Omega_1 \colon A \in \mathcal{F}\}$ and $\mu_1(A) = \mu(A)$, $A \in \mathcal{F}_1$, is σ-finite. Consider the induced decomposition $L^p(\mu) = L^p(\mu_1) \oplus L^p(\mu_2)$. Any positive linear surjective isometry on $L^p(\mu_1)$ extends (by identity on $L^p(\mu_2)$) to a positive linear surjective isometry on $L^p(\mu)$. It therefore suffices to show that the positive linear surjective isometries on $L^p(\mu_1)$ are not R-bounded.

It will be convenient and legitimate to interpret inclusions and equalities in $(\Omega_1, \mathcal{F}_1, \mu_1)$ modulo sets of measure zero. We adopt the formalism of [9] to view $(\Omega_1, \mathcal{F}_1, \mu_1)$ as a measure algebra rather than a measure space. If A and B are two distinct atoms in the measure algebra, then $A \cap B = \emptyset$. As $(\Omega_1, \mathcal{F}_1, \mu_1)$ is σ-finite, it contains at most countably many atoms, say B_i, $i \in N$, where $N \subset \mathbb{N}$. The sets $\Omega_3 = \bigcup_{i \in N} B_i$ and $\Omega_4 = \Omega_1 \setminus \Omega_3$ induce a decomposition $L^p(\mu_1) = L^p(\mu_3) \oplus L^p(\mu_4)$. The space $L^p(\mu_3)$ is isometrically isomorphic to $\ell^p(N)$ as a Banach lattice and the measure μ_4 has no atoms. If N is infinite, it is clear how to use Lemma 4.1 to show that the set of positive linear surjective isometries on $L^p(\mu_1)$ is not R-bounded.

If N is finite, then $\mu_4(\Omega_4) > 0$, as $L^p(\mu_1)$ is infinite-dimensional. Choose an $\Omega_5 \in \mathcal{F}_4$ with $0 < \mu_4(\Omega_5) < \infty$ and consider the induced decomposition $L^p(\mu_4) = L^p(\mu_5) \oplus L^p(\mu_6)$. Since $L^p(\mu)$ is separable, also $L^p(\mu_5)$ is separable and hence the measure space $(\Omega_5, \mathcal{F}_5, \mu_5)$ is separable (see for the definition [9, VIII.40, p. 168] and for the implication [9, VIII.42, p. 177]). From [9, VIII.41 Theorem C, p. 173] it follows that $(\Omega_5, \mathcal{F}_5, \mu_5)$ is as a measure algebra isomorphic to the interval $[0, \mu_5(\Omega_5)]$ with its Borel σ-algebra and Lebesgue measure. Thus, $L^p(\mu_5)$ is as a Banach lattice isometrically isomorphic to $L^p[0, \mu_5(\Omega_5)])$, and the proof can easily be completed with aid of Lemma 4.1. □

7. Two examples

The purpose of this section is to indicate how the previous results can be used for manipulations with resolvent families. We do not establish R-boundedness of the resolvent families mentioned in the Introduction and the examples serve as mere illustration. Let X be a Banach space of type $p > 1$ or of cotype $q < \infty$. For instance, $X = L^p(\mu)$, where $1 \leq p < \infty$ and $(\Omega, \mathcal{F}, \mu)$ is a measure space. Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on X with generator A such that $(0, \infty) \subset \rho(A)$.

Example 7.1. The following two statements are equivalent:

(a) $\{\lambda(\lambda I - A)^{-1} \colon \lambda \geq 1\}$ is R-bounded,
(b) $\{k^n(k^n I - A)^{-1} \colon k \in \mathbb{N},\ k \geq m\}$ is R-bounded for some $m, n \in \mathbb{N}$.

If we show the implication (b)⇒(a), then the equivalence is clear. We may assume that $m = 1$. We use that for a map $\lambda \mapsto S(\lambda) \colon [a, b] \to \mathcal{L}(X)$ with $\|S(\lambda) - S(\mu)\| \leq L|\lambda - \mu|$ for all $\lambda, \mu \in [a, b]$ and some constant L the set $\{S(\lambda) - S(a) \colon \lambda \in [a, b]\}$ is R_r-bounded by $L(b-a)$ for every $1 \leq r < \infty$. This follows readily from Proposition 2.1.1, 4, 5, and 7 and the observation that $\sum_{i=1}^n \|S(\lambda_i) - S(\lambda_{i-1})\| \leq L(b-a)$, so that the vector sum of the sets $\{0, S(\lambda_i) - S(\lambda_{i-1})\}$, $0 \leq i \leq n$, is R_r-bounded by $L(b-a)$, for any $a = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq b$.

Since there exists a constant $M \geq 0$ such that

$$\|\mu(\mu I - A)^{-1} - \lambda(\lambda I - A)^{-1}\| \leq (M/\lambda)|\lambda - \mu|, \quad \lambda, \mu \geq 1,$$

it follows that for each $k \in \mathbb{N}$,

$$\mathcal{T}_k := \{\lambda(\lambda I - A)^{-1} - k^n(k^n I - A)^{-1} \colon \lambda \in [k^n, k^{n+1}]\}$$

is R_r-bounded by $(M/k^n)((k+1)^n - k^n) \leq Mn((k+1)/k)^{n-1}/k$. By Theorem 3.1, it follows that $\bigcup_{k=1}^\infty \mathcal{T}_k$ is R-bounded, so that $\{\lambda(\lambda I - A)^{-1} \colon \lambda \geq 1\} \subset \{k^n(k^n I - A)^{-1} : k \in \mathbb{N}\} + \bigcup_{k=1}^\infty \mathcal{T}_k$ is R-bounded.

Example 7.2. If $\|T(t)\| \leq 1$ for all $t \geq 0$ and if the set of Cesaro means

$$\{n^{-1}(T(1) + \cdots + T(n)) \colon n \in \mathbb{N}\}$$

is R-bounded, then the set $\{\int_1^\infty \lambda e^{-\lambda t}T(t)\,\mathrm{d}t\colon\ \lambda \geq 0\}$ is R-bounded. Here $\int_1^\infty \lambda e^{-\lambda t}T(t)\,\mathrm{d}t$ denotes the bounded linear operator $x \mapsto \int_1^\infty \lambda e^{-\lambda t}T(t)x\,\mathrm{d}t$. For a proof, denote $S_n := n^{-1}\sum_{k=1}^n T(1)^k$, $n \in \mathbb{N}$.

We first show that the set $\{\sum_{k=1}^n \lambda e^{-\lambda k}T(k)\colon \lambda \geq 0,\ n \in \mathbb{N}\}$ is R-bounded. Indeed, according to [19, Lemma 2.2.6], the convex hull of the set of Cesaro means and the zero operator $\mathrm{co}\{0, S_k\colon\ k \in \mathbb{N}\}$ is also R-bounded. Now use that $\{\sum_{k=1}^n \alpha_k T(k)\colon\ n \in \mathbb{N},\ \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0,\ \sum_{k=1}^n \alpha_k \leq 1\} = \mathrm{co}\{0, S_k\colon\ k \in \mathbb{N}\}$. For a proof, notice for the less obvious inclusion that for $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0$ with $\alpha_1 + \cdots + \alpha_n \leq 1$ the choice $\beta_k := k(\alpha_k - \alpha_{k+1})$, $1 \leq k \leq n$, $(\alpha_{n+1} := 0)$ yields

$$\sum_{k=1}^n \beta_k \sum_{\ell=1}^k (1/k)T(\ell) = \sum_{\ell=1}^n \left(\sum_{k=\ell}^n (\alpha_k - \alpha_{k+1})\right) T(\ell) = \sum_{\ell=1}^n \alpha_\ell T(\ell)$$

and $\sum_{k=1}^n \beta_k \leq 1$.

Next, we show that $\{\int_1^n \lambda e^{-\lambda t}T(t)\,\mathrm{d}t\colon\ \lambda \geq 0,\ n \in \mathbb{N}\}$ is R-bounded. For a proof, observe first that $\lambda \mapsto \int_0^1 e^{-\lambda t}T(t)\,\mathrm{d}t$ is continuous and that for any $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{m+1}$,

$$\begin{aligned}\sum_{k=1}^m &\left\| \int_0^1 e^{-\lambda_{k+1}t}T(t)\,\mathrm{d}t - \int_0^1 e^{-\lambda_k t}T(t)\,\mathrm{d}t \right\| \\ &\leq \|T(1)\| \sum_{k=1}^m \int_0^1 (e^{-\lambda_k t} - e^{-\lambda_{k+1}t})\,\mathrm{d}t \\ &\leq \|T(1)\| \int_0^1 (e^{-\lambda_1} - e^{-\lambda_{m+1}t})\,\mathrm{d}t \leq \|T(1)\|.\end{aligned}$$

Hence $\lambda \mapsto \int_0^1 e^{-\lambda t}T(t)\,\mathrm{d}t$ is of bounded variation and thus $\{\int_0^1 e^{-\lambda t}T(t)\,\mathrm{d}t\colon\ \lambda \geq 0\}$ is R-bounded (use, e.g., [19, Theorem 2.2.8] or [7]). Further,

$$\begin{aligned}\sum_{k=1}^{n-1} &\lambda e^{-\lambda k}T(k) \int_0^1 e^{-\lambda t}T(t)\,\mathrm{d}t \\ &= \sum_{k=1}^{n-1} \int_0^1 \lambda e^{-\lambda(t+k)}T(t+k)\,\mathrm{d}t \\ &= \int_1^n \lambda e^{-\lambda t}T(t)\,\mathrm{d}t, \quad \lambda \geq 0,\ n \in \mathbb{N}.\end{aligned}$$

Due to Proposition 2.1.3 we obtain the asserted R-boundedness.

Finally, notice that $\{\int_1^\infty \lambda e^{-\lambda t}T(t)\,\mathrm{d}t\colon\ \lambda \geq 0\}$ is in the strong closure of $\{\int_1^n \lambda e^{-\lambda t}T(t)\,\mathrm{d}t\colon\ \lambda \geq 0,\ n \in \mathbb{N}\}$, so that it is an R-bounded set (see [19, Theorem 2.2.8] or [7]).

If $1 \in \rho(T(1))$, then the example of Section 3 yields that $\{S_k\colon\ k \in \mathbb{N}\}$ is R-bounded.

It remains to investigate whether the set $\{\int_0^1 \lambda e^{-\lambda t} T(t)\,\mathrm{d}t\colon \lambda \geq 0\}$ is R-bounded, in order to establish R-boundedness of the set

$$\{\lambda(\lambda I - A)^{-1} = \int_0^\infty \lambda e^{-\lambda t} T(t)\,\mathrm{d}t\colon \lambda \geq 1\}.$$

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On the Equation $\operatorname{div} u = g$ and Bogovskiĭ's Operator in Sobolev Spaces of Negative Order

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Dedicated to Philippe Clément on the occasion of his retirement

Abstract. Consider the divergence problem with homogeneous Dirichlet data on a Lipschitz domain. Two approaches for its solutions in the scale of Sobolev spaces are presented. The first one is based on Calderón-Zygmund theory, whereas the second one relies on the Stokes equation with inhomogeneous data.

Keywords. Divergence problem, Bogovskiĭ's operator.

1. Introduction

The solution of many problems in hydrodynamics requires a thorough understanding of the structure of the solutions of the equation $\operatorname{div} u = g$ for a given scalar valued function g. Hence, given a domain $\Omega \subset \mathbb{R}^n$, quite a few authors (see, e.g., [4, 12, 14, 18, 1, 2, 15, 17, 21, 3, 7, 8, 16]) dealt with the problem

$$\begin{cases} \operatorname{div} u = g \text{ in } \Omega \\ u_{|\partial\Omega} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

There are several approaches to prove the existence of a solution to problem (1.1), see [1, 8, 21, 15]. Observe also that the solution to this problem is not unique.

Bogovskiĭ proved the existence and a-priori estimates for a solution to (1.1) in the scale of Sobolev spaces of positive order provided $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, $n \geq 2$ and $g \in L^p(\Omega)$ satisfies $\int_\Omega g = 0$. Here $1 < p < \infty$. His approach is based on an explicit representation formula for u on star shaped domains. This representation of u as a singular integral allows to apply Calderón-Zygmund theory and estimates for u in Sobolev spaces of positive order follow thus by this theory.

In this paper we prove that Bogovskiĭ's solution operator B can be extended continuously to an operator acting from $W_0^{s,p}(\Omega)$ to $W_0^{s+1,p}(\Omega)^n$ provided $s >$

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$-2+1/p$. Our approach is based on properties of the adjoint kernel of K, K associated to B, see also [3, 13]. Results of this type are quite useful in the study of the Navier-Stokes flow past rotating obstacles, see e.g [9]. Note that the case $s=-1$ was already considered by Borchers and Sohr in [3]; see however [7] and the footnote on page 180 of Galdi's book [8].

A completely different approach to equation (1.1) is based on estimates for the solution of the inhomogeneous Stokes system

$$\begin{array}{rcll} -\Delta u + \nabla p &=& f & \text{in } \Omega \\ \operatorname{div} u &=& g & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \tag{1.2}$$

see Section 3. Setting $f=0$ and $Bg := u$, where (u,p) is the solution to problem (1.2), one obtains by this approach in particular estimates for the solution to problem (1.1) provided $g \in \widehat{W}^{1,p}(\Omega)$ or $g \in \widehat{W}^{-1,p}(\Omega)$. In Section 3 we extend this approach to $g \in \widehat{W}^{s,p}(\Omega)$ for all $s \in [-1,1]$. For related problems as, e.g., groundwater flow we refer to [5].

2. Approach by an explicit representation formula

Let $\Omega \subset \mathbb{R}^n$ be a domain and let $1<p<\infty$. For $s \geq 0$ $(W^{s,p}(\Omega), \|\cdot\|_{W^{s,p}(\Omega)})$ denote the usual Sobolev spaces, see, e.g., [20]. Furthermore, let $W_0^{s,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{s,p}(\Omega)}}$. For $s<0$ we set

$$W^{s,p}(\Omega) := (W_0^{-s,p'}(\Omega))' \text{ and } W_0^{s,p}(\Omega) := (W^{-s,p'}(\Omega))',$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Note that $C_c^\infty(\Omega)$ is dense in $W_0^{s,p}(\Omega)$ for all $s \in \mathbb{R}$.

Our first proposition relies on the fact that for bounded and star shaped domains with respect to a ball K a solution to problem (1.1) can be written as a singular integral. More precisely, choose $\omega \in C_c^\infty(K)$ with $\int_K \omega = 1$ and define for $g \in C_c^\infty(\Omega)$

$$(B_i g)(x) := \int_\Omega g(y) \frac{x_i - y_i}{|x-y|^n} \int_0^\infty \omega\Big(x + r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{n-1} \, \mathrm{d}r \, \mathrm{d}y, \tag{2.1}$$

where $x=(x_1,\dots,x_n)$ and $i=1,\dots,n$.

Then the following holds.

Proposition 2.1. *Let $1<p<\infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded and star shaped domain with respect to some ball. Let $g \in L^p(\Omega)$. Then $B := (B_1,\dots,B_n)$ satisfies*

$$BC_c^\infty(\Omega) \subset C_c^\infty(\Omega)^n$$

and

$$\nabla \cdot Bg = g - \omega \int_\Omega g \quad \textit{for } g \in L^p(\Omega).$$

Moreover, for $s > -2 + \frac{1}{p}$, B can be continuously extended to a bounded operator from $W_0^{s,p}(\Omega)$ to $W_0^{s+1,p}(\Omega)^n$.

Proof. The case $s \geq 0$ was already treated by Bogovskiĭ in [1]. There it is proved that $B_i C_c^\infty(\Omega) \subset C_c^\infty(\Omega)^n$. Moreover, by Calderón-Zygmund theory for the singular integral (2.1) one obtains the assertion for $s \geq 0$. For a detailed proof see [8, Lemma III.3.1].

In the following we prove the assertion for $s < 0$ by duality. In fact, the kernel of the adjoint B_i^* of the operator B_i is given by

$$\begin{aligned}
K_i^*(x, x-y) &= -\frac{x_i - y_i}{|x-y|^n} \int_0^\infty \omega(y - r\frac{x-y}{|x-y|^n})(|x-y| + r)^{n-1}\, \mathrm{d}r \\
&= -(x_i - y_i) \int_1^\infty \omega(x - r(x-y)) r^{n-1}\, \mathrm{d}r \\
&= -(x_i - y_i) \int_0^\infty \omega(x - r(x-y)) r^{n-1}\, \mathrm{d}r \ + \\
&\quad + (x_i - y_i) \int_0^1 \omega(x - r(x-y)) r^{n-1}\, \mathrm{d}r \\
&= -\frac{x_i - y_i}{|x-y|^n} \int_0^\infty \omega(x - r\frac{x-y}{|x-y|}) r^{n-1}\, \mathrm{d}r \ + \\
&\quad + (x_i - y_i) \int_0^1 \omega(x - r(x-y)) r^{n-1}\, \mathrm{d}r \\
&=: K_{i,\mathrm{sing}}^*(x, x-y) + K_{i,\mathrm{bdd}}^*(x, x-y).
\end{aligned}$$

Since

$$|\partial_{x_j} K_{i,\mathrm{bdd}}^*(x, x-y)| \leq C, \qquad i, j = 1, \ldots, n, x \in \Omega, y \in \mathbb{R}^n,$$

it follows that the operator associated to the kernel $K_{i,\mathrm{bdd}}^*$ continuously maps $L^p(\Omega)$ into $W^{1,p}(\Omega)$. We thus consider in the following the contribution of $K_{i,\mathrm{sing}}^*$. Similarly as in the proof of [8, Lemma III.3.1], the kernel $\partial_{x_j} K_{i,\mathrm{sing}}^*$ can be decomposed into a weakly singular kernel $K_{i,\mathrm{sing,w}}^*$ and a Calderón-Zygmund kernel $K_{i,\mathrm{sing,CZ}}^*$. More precisely, $\partial_{x_j} K_{i,\mathrm{sing}}^*$ can be rewritten as

$$\partial_{x_j} K_{i,\mathrm{sing}}^* = K_{i,\mathrm{sing,w}}^* + K_{i,\mathrm{sing,CZ}}^*,$$

with

$$\begin{aligned}
K_{i,\mathrm{sing,w}}^*(x, x-y) &= -\frac{x_i - y_i}{|x-y|^n} \int_0^\infty (\partial_{x_j}\omega)(x - r\frac{x-y}{|x-y|}) r^{n-1}\, \mathrm{d}r, \\
K_{i,\mathrm{sing,CZ}}^*(x, x-y) &= \frac{-\delta_{ij}}{|x-y|^n} \int_0^\infty \omega(x - r\frac{x-y}{|x-y|}) r^{n-1}\, \mathrm{d}r \\
&\qquad + \frac{x_i - y_i}{|x-y|^{n+1}} \int_0^\infty (\partial_{x_j}\omega)(x - r\frac{x-y}{|x-y|}) r^n\, \mathrm{d}r.
\end{aligned}$$

Note that $K^*_{i,\mathrm{sing,CZ}}$ satisfies the following properties

(a) $K^*_{i,\mathrm{sing,CZ}}(x,z) = \alpha^{-n} K^*_{i,\mathrm{sing,CZ}}(x,\alpha z)$, $x \in \Omega$, $z \in \mathbb{R}^n$, $\alpha > 0$,
(b) $\int_{|z|=1} K^*_{i,\mathrm{sing,CZ}}(x,z)\,\mathrm{d}z = 0$, $x \in \Omega$,
(c) $|K^*_{i,\mathrm{sing,CZ}}(x,z)| \le C$, $x \in \Omega$, $|z| = 1$.

It follows from classical Calderón-Zygmund theory [6, 19] that

$$B_i^* \in \mathcal{L}(L^p(\Omega), W^{1,p}(\Omega)).$$

Moreover,

$$B_i^* \in \mathcal{L}(W_0^{\tilde{s},p}(\Omega), W^{\tilde{s}+1,p}(\Omega)), \quad \tilde{s} > 0$$

by [8, Remark III.3.1] and real interpolation. Since $W_0^{s,p}(\Omega) = W^{s,p}(\Omega)$ for $-1 + \frac{1}{p} < s < \frac{1}{p}$, we obtain

$$B \in \mathcal{L}(W_0^{s,p}(\Omega), W_0^{s+1,p}(\Omega)^n), \qquad -2 + \frac{1}{p} < s < -1.$$

The remaining cases finally follow by real interpolation. □

Remark 2.2.

(a) It should be emphasized that in the above proposition, the operator B is defined for all $g \in L^p(\Omega)$ whereas Bogovskiĭ [1], von Wahl [21] and Galdi [8] constructed solutions to the problem (1.1). The latter is only possible if $\int_\Omega g = 0$. Hence, B may be regarded as extension of the solution operator to problem (1.1). However, if $\int_\Omega g \ne 0$, then Bg is not a solution to (1.1).
(b) The idea of using the adjoint kernel to prove estimates for B in Sobolev spaces of negative order is quite natural and was already used in [3] for the case $s = -1$. More recently, this approach was also reconsidered by [13].
(c) There is a considerable difference between B_i and its adjoint B_i^*. As $B_i C_c^\infty(\Omega) \subset C_c^\infty(\Omega)$, this does not hold true for its adjoint.
(d) The above proof shows that $B \in \mathcal{L}(W_0^{s,p}(\Omega), W^{s+1,p}(\Omega)^n)$ for $s \le -1$.

Bounded and locally Lipschitz domains have the remarkable property that they can be written as a finite union of star shaped domains. This gives us the possibility to carry over mapping properties of the operator B, originally defined on star shaped domains, to locally Lipschitz domains. For convenience and to fix notation we first state a result concerning the decomposition of Lipschitz domains into star shaped domains. For a proof of this result we refer to [8, Lemma III.3.4].

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and locally Lipschitz domain. Then there exist $m \in \mathbb{N}$ and an open cover $\mathcal{G} = \{G_i : i \in \{1,\dots,m\}\}$ of $\overline{\Omega}$ such that for $1 \le i \le m$ the set $\Omega_i = \Omega \cap G_i$ is star shaped with respect to some ball and $\Omega = \bigcup_{i=1}^m \Omega_i$.*

Moreover, there exist $\phi_i \in C_c^\infty(G_i)$, $m_i \in \mathbb{N}$, $\theta_{i,k} \in C_c^\infty(\Omega_i)$ and $\psi_{i,k} \in C_c^\infty(\overline{\Omega})$ ($i \in \{1,\dots,m\}$, $k \in \{1,\dots,m_i\}$) such that

$$P_i g := \phi_i g + \sum_{k=1}^{m_i} \theta_{i,k} \int_\Omega \psi_{i,k} g, \quad g \in C_c^\infty(\Omega)$$

satisfies $P_i g \in C_c^\infty(\Omega_i)$ *and* $\int_\Omega P_i g = 0$. *In addition, if* $\int_\Omega g = 0$ *we get a decomposition of* g *by* $g = \sum_{i=1}^m P_i g$.

In order to define a solution operator to (1.1) for bounded, locally Lipschitz domains we reconsider the operators P_i.

Lemma 2.4. *Let* $1 < p < \infty, s \in \mathbb{R}$ *and let* Ω_i *and* P_i *be defined as in Lemma* 2.3 *for* $i = 1, \dots, m$. *Then*

$$P_i \in \mathcal{L}(W_0^{s,p}(\Omega), W_0^{s,p}(\Omega_i)),\ s \in \mathbb{R},\ 1 < p < \infty.$$

Proof. Let $i \in \{1, \dots, m\}$. Consider first the case where $s \geq 0$. Then there exists $C > 0$ such that

$$\|P_i g\|_{W_0^{s,p}(\Omega_i)} \leq C\|g\|_{W_0^{s,p}(\Omega)}, \quad g \in W_0^{s,p}(\Omega).$$

In order to prove the remaining cases, let $s > 0$, $g \in C_c^\infty(\Omega)$ and $v \in W^{s,p'}(\Omega_i)$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\begin{aligned}
|\langle P_i g, v\rangle| &= \left|\int_{\Omega_i} \phi_i(x)g(x)v(x) + \sum_{k=1}^m \theta_{i,k}(x)v(x)\int_\Omega \psi_{i,k}(y)g(y)\ \mathrm{d}y\ \mathrm{d}x\right| \\
&= \left|\int_{\Omega_i} g(x)\phi_i(x)v(x)\ \mathrm{d}x + \sum_{k=1}^m \int_\Omega \psi_{i,k}(y)g(y)\int_\Omega \theta_{i,k}(x)v(x)\ \mathrm{d}x\ \mathrm{d}y\right| \\
&\leq \|g\|_{W_0^{-s,p}(\Omega)}\|\phi_i v\|_{W^{s,p'}(\Omega)} + \|g\|_{W_0^{-s,p}(\Omega)}\sum_{k=1}^m \left\|\psi_{i,k}\int_\Omega \theta_{i,k}(x)v(x)\ \mathrm{d}x\right\|_{W^{s,p'}(\Omega_i)} \\
&\leq C\|g\|_{W_0^{-s,p}(\Omega)}\|v\|_{W^{s,p'}(\Omega_i)}
\end{aligned}$$

where C is some constant independent of g and v. $\square$

The following theorem is the main result of this paper. Besides its interest in its own, there are many applications of the use of Bogovskiĭ's operator in Sobolev spaces of negative order; see, e.g., the recent paper [9].

Theorem 2.5. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with a locally Lipschitz boundary. Then there exists* $B : C_c^\infty(\Omega) \to C_c^\infty(\Omega)^n$ *such that*

$$\nabla \cdot Bg = g, \quad g \in L^p(\Omega) \ \textit{with} \int_\Omega g = 0.$$

Moreover, B *can be extended continuously to a bounded operator from* $W_0^{s,p}(\Omega)$ *to* $W_0^{s+1,p}(\Omega)^n$ *provided* $s > -2 + \frac{1}{p}$.

Proof. Let $g \in C_c^\infty(\Omega)$. Consider the decomposition of Ω and the associated operators P_i defined as in Lemma 2.3. Then

$$\sum_{i=1}^m P_i g = g \text{ provided } \int_\Omega g(x)\ \mathrm{d}x = 0. \tag{2.2}$$

Let B_i denote the operator defined in Proposition 2.1 acting on Ω_i and set $Bg := \sum_{i=1}^m B_i P_i g$. Then by Proposition 2.1 and Lemma 2.4,

$$B \in \mathcal{L}(W_0^{s,p}(\Omega), W_0^{s+1,p}(\Omega)^n) \text{ for all } s \geq 0$$

since $B_i P_i C_c^\infty(\Omega) \subset C_c^\infty(\Omega_i)^n$.

Again, by Proposition 2.1 and Lemma 2.4

$$\begin{aligned}
|\langle Bg, v\rangle| = \left| \int_\Omega \sum_{i=1}^m B_i P_i g(x) v(x) \, \mathrm{d}x \right| &= \left| \sum_{i=1}^m \int_{\Omega_i} B_i P_i g(x) v(x) \, \mathrm{d}x \right| \\
&\leq C \sum_{i=1}^m \|P_i g\|_{W_0^{-s,p}(\Omega_i)} \|v\|_{W^{s,p'}(\Omega_i)} \\
&\leq C \|g\|_{W_0^{-s,p}(\Omega)} \|v\|_{W^{s,p'}(\Omega)}, \quad g \in C_c^\infty(\Omega), \ v \in W^{s,p'}(\Omega).
\end{aligned}$$

Finally, by (2.2) we obtain

$$\nabla \cdot Bg = \sum_{i=1}^m \nabla \cdot B_i P_i g = \sum_{i=1}^m P_i g = g, \quad g \in L^p(\Omega), \ \int_\Omega g(x) \, \mathrm{d}x = 0. \qquad \square$$

3. Approach by the inhomogeneous Stokes equation

We start this section by considering the problem

$$\begin{aligned}
-\Delta u + \nabla p &= f && \text{in } \Omega \\
\operatorname{div} u &= g && \text{in } \Omega \\
u &= 0 && \text{on } \partial\Omega
\end{aligned} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^n$ for $n \geq 2$ is a bounded domain with boundary $\partial\Omega \in C^2$. Let $1 < p, p' < \infty$ such that $1 = \frac{1}{p} + \frac{1}{p'}$.

We then set $L_0^p(\Omega) := \{f \in L^p(\Omega) : \int_\Omega f = 0\}$ and for $s \in [0,1]$ let $\widehat{W}^{s,p}(\Omega) := W^{s,p}(\Omega) \cap L_0^p(\Omega)$ equipped with the norm in $W^{s,p}(\Omega)$. Furthermore, we define $\widehat{W}^{-s,p}(\Omega) := (\widehat{W}^{s,p'}(\Omega))'$ equipped with the usual dual norm.

Note that for $g \in \widehat{W}^{1,p}(\Omega)$ we have

$$\|g\|_{\widehat{W}^{-1,p}(\Omega)} = \sup_{v \in \widehat{W}^{1,p}(\Omega) \setminus \{0\}} \frac{|\langle g, v\rangle|}{\|v\|_{W^{1,p'}(\Omega)}} \leq \|g\|_{W_0^{-1,p}(\Omega)}.$$

The following proposition is due to Farwig and Sohr [7].

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. Let $1 < p < \infty$. Then there exists a bounded operator $R : \widehat{W}^{1,p}(\Omega) \to W^{2,p}(\Omega)^n \cap W_0^{1,p}(\Omega)^n$ such that $\operatorname{div} Rg = g$. Moreover, R satisfies the following estimates:*

(a) $\|Rg\|_{L^p(\Omega)} \leq C\|g\|_{\widehat{W}^{-1,p}(\Omega)} \leq C\|g\|_{W_0^{-1,p}(\Omega)}$,

(b) $\|Rg\|_{W^{2,p}(\Omega)} \leq C\|g\|_{W^{1,p}(\Omega)}$.

Here $C > 0$ is a constant depending on Ω and p only.

Setting $f = 0$ and $Rg := u$, where (u, p) is the solution of (3.1) the assertion above is a direct consequence of the unique solvability of the problem (3.1) with $f \in (L^p(\Omega))^n$ and $g \in \widehat{W}^{1,p}(\Omega)$.

Similarly, we obtain the fact that $R \in \mathcal{L}(L^p_0(\Omega), (W^{1,p}(\Omega))^n)$ from the unique solvability of the problem (3.1) for $f \in W^{-1,p}(\Omega)^n$ and $g \in L^p_0(\Omega)$. In fact, since (u, p) is the unique solution of (3.1), the operator R given in Proposition 3.1 may be thus extended from $\widehat{W}^{1,p}(\Omega)$ to $L^p_0(\Omega)$. Unique solvability of (3.1) in the given setting was first proved by Cattabriga [4] for the case $n = 3$ and by Galdi and Simader [10] for general $n \geq 2$. See also [11] for a different proof. We summarize these facts in the next proposition.

Proposition 3.2. *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. Then for every $f \in W^{-1,p}(\Omega)^n$ and $g \in L^p_0(\Omega)$ there exists a unique solution $(u, p) \in W^{1,p}_0(\Omega)^n \times L^p_0(\Omega)$ of (3.1) satisfying the inequality*

$$\|\nabla u\|_{L^p(\Omega)} + \|p\|_{L^p(\Omega)} \leq C(\|f\|_{W^{-1,p}(\Omega)^n} + \|g\|_{L^p(\Omega)})$$

for some constant $C = C(\Omega, n, p)$.

Noting that $(L^{p'}_0(\Omega))' = L^p_0(\Omega)$, the following lemma implies that $L^p_0(\Omega)$ is dense in $\widehat{W}^{-1,p}(\Omega)$. The proof is standard and therefore omitted.

Lemma 3.3. *Let X, Y be Banach spaces. Assume that X is densely embedded in Y and that X is reflexive. Then the closure of Y' is X'.*

Combining the above results, we may define a solution operator for the divergence problem (1.1) in the following spaces

$$R : \begin{cases} \widehat{W}^{1,p}(\Omega) & \to \quad W^{2,p}(\Omega)^n \cap W^{1,p}_0(\Omega)^n \\ L^p_0(\Omega) & \to \quad W^{1,p}_0(\Omega)^n \\ \widehat{W}^{-1,p}(\Omega) & \to \quad L^p(\Omega)^n \end{cases}$$

The following result gives additional mapping properties of R in the scale of Sobolev spaces.

Theorem 3.4. *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. Let $s \in [-1, 1]$. Then there exists a bounded linear operator $R : \widehat{W}^{s,p}(\Omega) \to W^{s+1,p}(\Omega)^n$ such that $\operatorname{div} Rg = g$ for all $g \in \widehat{W}^{s,p}(\Omega)$.*

Proof. The cases $s = 1$ and $s = -1$ follow from Proposition 3.1. Consider next the case where $0 \leq s < 1$. Let K denote the set of all constant functions over Ω. Then we may identify the spaces $L^p_0(\Omega)$ with $L^p(\Omega)/K$ and $\widehat{W}^{1,p}(\Omega)$ with $W^{1,p}(\Omega)/K$, respectively. As K is a one-dimensional vector space, it follows from [20, Section 1.17.2, Remark 1] that

$$\begin{aligned} (L^p_0(\Omega), \widehat{W}^{1,p}(\Omega))_{s,p} &= (L^p(\Omega)/K, W^{1,p}(\Omega)/K)_{s,p} \\ &= (L^p(\Omega), W^{1,p}(\Omega))_{s,p}/K = \widehat{W}^{s,p}(\Omega). \end{aligned}$$

This implies the assertion provided $0 \leq s < 1$.

In order to prove the remaining cases where $-1 < s < 0$, note that

$$(L_0^{p'}(\Omega), \widehat{W}^{1,p'}(\Omega))'_{s,p'} = (L_0^p(\Omega), \widehat{W}^{-1,p}(\Omega))_{s,p};$$

see, e.g., [20, Section 1.11.2]. Hence,

$$\widehat{W}^{-s,p}(\Omega) = (L_0^{p'}(\Omega), \widehat{W}^{1,p'}(\Omega))'_{s,p'} = (L_0^p(\Omega), \widehat{W}^{-1,p}(\Omega))_{s,p}$$

and the proof is complete. □

Remark 3.5. The assertions of Theorems 2.5 and 3.4 remain valid also for the complex interpolation spaces. In fact, the above mapping properties of B and R in the scale of Sobolev spaces hold true also in the scale of the spaces $H^{s,p}(\Omega)$ and $\widehat{H}^{s,p}(\Omega)$, respectively.

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Renormalization of Interacting Diffusions: A Program and Four Examples

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Dedicated to Philippe Clément on the occasion of his retirement

Abstract. Systems of hierarchically interacting diffusions allow for a rigorous *renormalization analysis*. By bringing into play the powerful machinery of stochastic analysis, it is possible to obtain a complete classification of the large space-time behavior of these systems into *universality classes*. The present paper outlines a general renormalization program that is being pursued since ten years and describes four examples where this program has been successfully carried through. The systems under consideration model the evolution of multi-type populations subject to migration and resampling.

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1. Overview

1.1. Interacting diffusions

The systems that will be considered in this paper evolve according to the following set of coupled one-dimensional SDE's:

$$dX_\eta(t) = \sum_{\zeta\in\Omega_N} a_N(\eta,\zeta)\,[X_\zeta(t)-X_\eta(t)]\,dt + \sqrt{g(X_\eta(t))}\,dW_\eta(t), \qquad \eta\in\Omega_N,\ t\ge 0, \tag{1.1}$$

where

(1) $X_\eta(t)\in S\subseteq\mathbb{R}$ is the single-component *state space.*
(2) Ω_N is the *hierarchical group* of order $N\in\mathbb{N}$.
(3) $a_N(\cdot,\cdot)$ is the *interaction kernel* on $\Omega_N\times\Omega_N$.
(4) $g(\cdot)$ is the $[0,\infty)$-valued *diffusion function* on S.
(5) $\{W_\eta(\cdot)\}_{\eta\in\Omega_N}$ are independent standard Brownian motions on $\mathbb{R}$.

We will also look at higher-dimensional versions of (1.1). In what follows we will focus on one particular choice for $a_N(\cdot,\cdot)$, but we will consider several choices of S and g. As *initial condition* we take

$$X_\eta(0) = \theta \in \mathrm{int}(S) \quad \forall\, \eta \in \Omega_N. \tag{1.2}$$

Equation (1.1) arises as the continuum limit of discrete models in *population dynamics*. In these models, individuals of different types live in large *colonies*, labelled by the hierarchical group. The state of a colony describes the *composition* of the population at that colony (such as the fractions or the total masses of the different types of individuals). Individuals *migrate* between colonies, i.e., they move from one colony to another according to a random mechanism that depends on the locations of the two colonies. This is described by the first term in the right-hand side of (1.1). Moreover, individuals are subject to *resampling* within each colony, i.e., they are replaced by new individuals according to a random mechanism that depends on the state of the colony. This is described by the second term in the right-hand side of (1.1). The system in (1.1) arises after letting the number of individuals per colony tend to infinity and normalizing both the state and the rate of evolution of the colony appropriately. For more background, the reader is referred to Sawyer and Felsenstein [20] and Ethier and Kurtz [16], Chapter 10.

1.2. Hierarchical group and multi-scale block averages

The hierarchical group of order N is the set

$$\Omega_N = \left\{ \xi = (\xi_i)_{i\in\mathbb{N}} \in \{0,1,\ldots,N-1\}^{\mathbb{N}} \colon \sum_{i\in\mathbb{N}} \xi_i < \infty \right\} \tag{1.3}$$

with addition modulo N. On Ω_N, the hierarchical distance is defined as

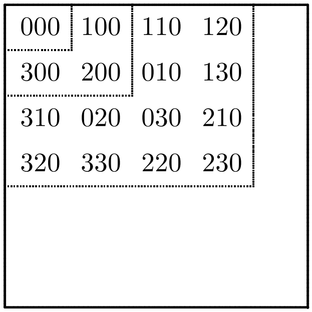
$$d(\eta,\zeta) = \min\{i \in \mathbb{N}\cup\{0\} \colon \eta_j = \zeta_j \ \forall\, j > i\}, \tag{1.4}$$

which is an *ultrametric*.

At first sight, the choice for Ω_N as the index set for the colonies may seem a bit artificial. However, it is completely natural within a genetics context. Indeed, what Ω_N does is organize the colonies according to their *genetic location*. The population is divided into families, clans, neighborhoods, villages, regions, etc. Site ξ contains all individuals that are in family ξ_1, clan ξ_2, neighborhood ξ_3, village ξ_4, region ξ_5, etc. The hierarchical distance between two colonies measures the smallest level in the hierarchy to which both colonies belong.

Our goal will be to study the evolution of the system in (1.1–1.2) on large space-time scales in the limit as $N \to \infty$, the so-called *hierarchical mean-field limit*. To that end, we define *block averages on space-time scale* $k \in \mathbb{N}\cup\{0\}$ by putting

$$Y^{[k]}_{\eta,N}(t) = \frac{1}{N^k} \sum_{\substack{\zeta\in\Omega_N \\ d(\eta,\zeta)\le k}} X_\zeta(N^k t), \qquad \eta \in \Omega_N,\ t \ge 0, \tag{1.5}$$



Schematic picture of Ω_4. Blocks of radius 0, 1 and 2 around the origin (first three digits indicated only).

where we average over all components in a block of radius k around a given site and speed up time proportionally to the size of the block.

In what follows, we will make a particular choice for the interaction kernel in (1.1), namely,

$$a_N(\eta,\zeta) = \sum_{k\geq d(\eta,\zeta)} c_k N^{1-2k} \quad \forall\, \eta \neq \zeta, \tag{1.6}$$

where $(c_k)_{k\in\mathbb{N}}$ is a given sequence of positive constants such that the sum is finite for all N large enough. One reason for this choice is that (1.1) takes on a particularly suitable form,

$$dX_\eta(t) = \sum_{k\geq 1} c_k N^{1-k} \left[Y^{[k]}_{\eta,N}(tN^{-k}) - X_\eta(t) \right] dt + \sqrt{g(X_\eta(t))}\, dW_\eta(t), \tag{1.7}$$

where single components are attracted towards successive block averages. Another reason is that, modulo normalization, (1.6) is the transition kernel of a random walk on Ω_N whose potential-theoretic properties are independent of N, for N sufficiently large, and are completely determined by the sequence $(c_k)_{k\in\mathbb{N}}$.

The interpretation of (1.6) is that the random walk picks an integer k with probability proportional to c_k/N^{k-1} and jumps to a site within the k-block around its current position according to the uniform distribution on this k-block. Throughout the paper we will assume that

$$\sum_{k\in\mathbb{N}} \frac{1}{c_k} = \infty, \tag{1.8}$$

which makes the random walk on Ω_N *critically recurrent.* This is crucial for the universal behavior to be described later on.

The key point about (1.7) is that it is susceptible to a *renormalization analysis.*

1.3. Renormalization transformation

Let us consider the blocks around the origin. If we let $N \to \infty$ in (1.7), then only the term with $k = 1$ survives. Moreover, the term $Y^{[1]}_{0,N}(tN^{-1})$ converges to θ for all $t \geq 0$, because of (1.2). Therefore we find that ($\Longrightarrow$ denotes convergence in law)

$$\{X_0(t)\colon\ t \geq 0\} \Longrightarrow \{Z_{\theta,g,c_1}(t)\colon\ t \geq 0\} \qquad \text{as } N \to \infty, \tag{1.9}$$

where $Z_{\theta,g,c_1}(t)$ is the solution of the *autonomous* SDE

$$dZ(t) = c_1\,[\theta - Z(t)]\,dt + \sqrt{g(Z(t))}\,dW(t), \qquad Z(0) = \theta. \tag{1.10}$$

In other words, in the limit as $N \to \infty$ the single components *decouple* and follow a simple diffusion equation parameterized by θ, g and c_1. (The behavior expressed by (1.9–1.10) is often referred to as "McKean-Vlasov limit" and "propagation of chaos".) Under mild restrictions on S and g, this diffusion equation has an *equilibrium distribution*, which we denote by ν_{θ,g,c_1} and which lives on S.

We next move up one step in the hierarchy. By summing (1.7) over the components in a 1-block, we get

$$\begin{aligned} dY^{[1]}_{\eta,N}(t) = &\sum_{k\geq 2} c_k N^{2-k} \left[Y^{[k]}_{\eta,N}(tN^{1-k}) - Y^{[1]}_{\eta,N}(t)\right]\, dt \\ &+ \frac{1}{\sqrt{N}} \sum_{\substack{\zeta\in\Omega_N \\ d(\eta,\zeta)\leq 1}} \sqrt{g(X_\zeta(Nt))}\,dW_\zeta(t). \end{aligned} \tag{1.11}$$

Here, time is scaled up by a factor N, both in the 1-block and in the Brownian motions (hence the factor $1/\sqrt{N}$), and the term with $k = 1$ cancels out. If we let $N \to \infty$ in (1.11), then only the term with $k = 2$ survives. Moreover, the term $Y^{[2]}_{0,N}(tN^{-1})$ converges to θ for all $t \geq 0$ because of (1.2). Furthermore, if $Y^{[1]}_{0,N}(t) = y$, then each of the N components in this 1-block is linearly attracted towards a value that is approximately y, since the attraction towards the values of the k-blocks with $k \geq 2$ is weak when N is large (recall (1.7)). Therefore, at time Nt each of these components is close in distribution to the equilibrium ν_{y,g,c_1} (associated with (1.10) after replacing θ by y). Thus, it is *reasonable* to expect that

$$\left\{Y^{[1]}_{0,N}(t)\colon\ t \geq 0\right\} \Longrightarrow \left\{Z_{\theta,F_{c_1}g,c_2}(t)\colon\ t \geq 0\right\} \qquad \text{as } N \to \infty, \tag{1.12}$$

where $Z_{\theta,F_{c_1}g,c_2}(t)$ is the solution of the SDE

$$dZ(t) = c_2\,[\theta - Z(t)]\,dt + \sqrt{(F_{c_1}g)(Z(t))}\,dW(t), \qquad Z(0) = \theta, \tag{1.13}$$

where $F_{c_1}g$ is the *diffusion function on scale* 1 obtained from the diffusion function g on scale 0 by *averaging* it w.r.t. the equilibrium distribution associated with (1.10) on scale 0:

$$(F_{c_1}g)(y) = \int_S g(x)\nu_{y,g,c_1}(dx). \tag{1.14}$$

This formula defines a *renormalization transformation* F_{c_1} acting on the function g.

The above renormalization procedure can be iterated. Indeed, it is *reasonable* to expect that

$$\left\{Y^{[k]}_{0,N}(t)\colon\ t \geq 0\right\} \Longrightarrow \left\{Z_{\theta,F^{[k]}g,c_{k+1}}(t)\colon\ t \geq 0\right\} \qquad \text{as } N \to \infty, \tag{1.15}$$

where $Z_{\theta,F^{[k]}g,c_{k+1}}(t)$ is the solution of the SDE

$$dZ(t) = c_{k+1}\,[\theta - Z(t)]\,dt + \sqrt{(F^{[k]}g)(Z(t))}\,dW(t), \qquad Z(0) = \theta, \tag{1.16}$$

where $F^{[k]}g$ is the *diffusion function on scale* k obtained from the diffusion function g on scale 0 by applying k times the renormalization transformation:

$$F^{[k]} = F_{c_k} \circ \cdots \circ F_{c_1}. \tag{1.17}$$

The intuition behind this claim is as follows:

(a) On time scale $N^k t$, the block averages on scale k *fluctuate* while the block averages on scales $> k$ *almost stand still* and therefore remain close to the initial value θ.
(b) Given that the block averages on scale k have value y, the block averages on scale $k-1$ reach *equilibrium* with drift towards y almost instantly on time scale $N^k t$.
(c) Consequently, the diffusion function on scale k is the *average* of the diffusion function on scale $k-1$ under this equilibrium.

The limit $N \to \infty$ provides the *separation of successive space-time scales.*

1.4. Renormalization program

The above heuristic observations naturally lead to a two-step programme for renormalization:

(I) **Stochastic part**: Show that (1.15–1.16) indeed arise from (1.1–1.2) in the limit as $N \to \infty$ for all scales $k \in \mathbb{N}$.
(II) **Analytic part**: Study the orbits of $(F^{[k]})_{k\in\mathbb{N}}$, determine their fixed points, and classify their domains of attraction.

The goal is to try and carry out this programme for relevant choices of S for appropriate classes $\mathcal{H} = \mathcal{H}(S)$ of diffusion functions. For tutorial overviews on this programme, see Greven [17] and den Hollander [18].

In what follows we will describe two *one-dimensional* examples where the above renormalization program has been *fully* carried through (Section 2) and two *higher-dimensional* examples where it has been *partially* carried through (Section 3). It will turn out that, in each of these four examples, $(F^{[k]})_{k\in\mathbb{N}}$ has an interesting structure of fixed points and domains of attraction. The fixed points correspond to *special choices* of g that play the role of *universal attractors* for the dynamics (1.1–1.2) on the *macroscopic scale* corresponding to $k \to \infty$. We close with listing some open problems (Section 4).

For ease of exposition, we will henceforth restrict to the case where $c_k = 1$ for all k. We then have $F^{[k]} = F^k$ with $F = F_1$. It is straightforward to extend the results to the case (1.8).

2. One dimension

2.1. $S = [0, 1]$

In this example, our choice for the state space and the class of diffusion functions is:

$S = [0, 1]$ and $g \in \mathcal{H}$, the class of functions satisfying:

1. g is Lipschitz on $[0, 1]$.
2. $g(x) > 0$ for $x \in (0, 1)$.
3. $g(0) = g(1) = 0$.

The *stochastic part* of the renormalization program was carried out by Dawson and Greven [9], [10] and is given in Theorem 2.1.

Theorem 2.1. *In the limit as* $N \to \infty$, (1.15–1.16) *arise from* (1.1–1.2) *with* F *given by*

$$(Fg)(y) = \int_{[0,1]} g(x) \nu_{y,g}(dx), \tag{2.1}$$

where $\nu_{y,g}$ *is the equilibrium distribution of*

$$dZ(t) = [y - Z(t)]\, dt + \sqrt{g(Z(t))}\, dW(t), \tag{2.2}$$

which is given by

$$\nu_{y,g}(dx) = \frac{1}{Z_{y,g}} \frac{1}{g(x)} \exp\left[- \int_y^x \frac{z - y}{g(z)}\, dz \right] dx \tag{2.3}$$

with $Z_{y,g}$ *the normalizing constant.*

Note that F is an integral operator. Since $\nu_{y,g}$ depends on g itself, F is non-linear.

The *analytic part* of the renormalization program was carried out by Baillon, Clément, Greven and den Hollander [2] and is given in Theorems 2.2–2.4.

Theorem 2.2. (a) $F\mathcal{H} \subset \mathcal{H}$.
(b) $\forall\, g \in \mathcal{H}$*:* $y \mapsto (Fg)(y)$ *is* C^∞ *on* $(0, 1)$.

Theorem 2.3. *The solution of the eigenvalue problem* $Fg = \lambda g$, $g \in \mathcal{H}$, $\lambda > 0$, *is the* 1*-parameter family*

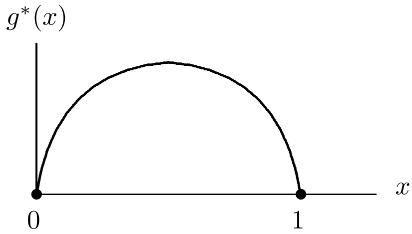
$$g = dg^* \ \textit{and}\ \lambda = \frac{1}{1 + d}, \qquad d > 0, \tag{2.4}$$

where $g^*(x) = x(1 - x)$.

Theorem 2.4. *For all* $g \in \mathcal{H}$,

$$\lim_{k \to \infty} kF^k g = g^* \tag{2.5}$$

uniformly on $[0, 1]$.



Fisher-Wright diffusion function.

These results show that F is well defined on the class $\mathcal{H}$, is smoothing, and has a *fixed shape* g^* that is *globally attracting* after proper normalization. Thus, (1.1–1.2) exhibits *full universality*: No matter what the diffusion function g on scale 0 is, the diffusion function $F^k g$ on scale k is close to $(1/k)g^*$. The case $g = g^*$ is called the *Fisher-Wright diffusion.*

2.2. $S = [0, \infty)$

In this example, our choice for the state space and the class of diffusion functions is:

$S = [0, \infty)$ and $g \in \mathcal{H}$, the class of functions satisfying:

1. g is locally Lipschitz on $[0, \infty)$.
2. $g(x) > 0$ for $x > 0$.
3. $g(0) = 0$.
4. $\lim_{x\to\infty} g(x)/x^2 = 0$.

The *stochastic part* of the renormalization program was carried out by Dawson and Greven [11]. The same formulas as in Theorem 2.1 apply, but now on $[0, \infty)$ instead of $[0, 1]$.

The *analytic part* of the renormalization program was carried out by Baillon, Clément, Greven and den Hollander [3] and is given in Theorems 2.5–2.8.

Theorem 2.5. (a) $F\mathcal{H} \subset \mathcal{H}$.
(b) $\forall\, g \in \mathcal{H}$*:* $y \mapsto (Fg)(y)$ *is* C^∞ *on* $(0, \infty)$*.*

Theorem 2.6. *The solution of the eigenvalue problem* $Fg = \lambda g$*,* $g \in \mathcal{H}$*,* $\lambda > 0$*, is the* 1*-parameter family*

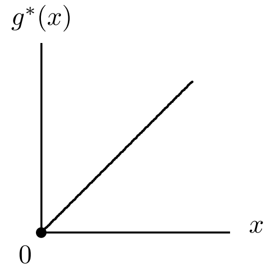
$$g = dg^* \text{ and } \lambda = 1, \qquad d > 0, \tag{2.6}$$

where $g^*(x) = x$*.*

Theorem 2.7. *If* $\lim_{x\to\infty} x^{-1} g(x) = d$*, then*

$$\lim_{k\to\infty} F^k g = dg^* \tag{2.7}$$

uniformly on compact subsets of $[0, \infty)$*.*



Feller diffusion function.

Theorem 2.8. *Suppose that* $g(x) \sim x^\alpha L(x)$ *as* $x \to \infty$ *with* $\alpha \in (0,2)\backslash\{1\}$ *and* L *slowly varying at infinity. Let* $(e_k)_{k\in\mathbb{N}}$ *be defined by*

$$\frac{1}{k} = \frac{g(e_k)}{e_k^2}. \tag{2.8}$$

Then there exist constants $0 < K_1(\alpha) \le K_2(\alpha) < \infty$ *such that*

$$K_1(\alpha)g^* \le \liminf_{k\to\infty} \frac{k}{e_k} F^k g \le \limsup_{k\to\infty} \frac{k}{e_k} F^k g \le K_2(\alpha)g^* \tag{2.9}$$

uniformly on compact subsets of $[0,\infty)$.

Remark 2.9.

(i) Theorem 2.7 (which is Theorem 2.8 for $\alpha = 1$ and $L \equiv d$) says that all g that are asymptotically linear are iterated towards a linear with the same slope.

(ii) In Theorem 2.8, if $L \equiv 1$, then $e_k \sim k^{1/(2-\alpha)}$ as $k \to \infty$, and so

$$F^k g \asymp k^{-(1-\alpha)/(2-\alpha)} g^*. \tag{2.10}$$

Thus, concave g are iterated downwards, while convex g are iterated upwards.

(iii) It is shown in [3] that all solutions of (2.8) have the same asymptotic behavior as $k \to \infty$.

(iv) It is conjectured in [3] that

$$K_1(\alpha) = K_2(\alpha) = (\alpha!)^{1/(2-\alpha)} 2^{(1-\alpha)/(2-\alpha)}. \tag{2.11}$$

The above results show that g^* again acts as a *globally attracting fixed point*, except that now the *normalization* depends on the behavior of g at infinity. Thus, once again (1.1–1.2) exhibits *universality*: No matter what the diffusion function g on scale 0 is, the diffusion function $F^k g$ on scale k is close to $(e_k/k)g^*$. The case $g = g^*$ is called the *Feller diffusion*.

3. Higher dimension

In higher dimension our system in (1.1) must be written in vector form:

$$dX_\eta^i(t) = \sum_{\zeta \in \Omega_N} a_N(\eta,\zeta)\ \left[X_\zeta^i(t) - X_\eta^i(t)\right]\ dt + \sqrt{g^i\left(\vec{X}_\eta(t)\right)}\ dW_\eta^i(t), \quad 1 \le i \le d. \tag{3.1}$$

Here, the scalar component $X_\eta(t)$ in (1.1) is replaced by the vector

$$\vec{X}_\eta(t) = \left(X_\eta^1(t), \ldots, X_\eta^d(t)\right) \in S \subseteq \mathbb{R}^d,$$

while the scalar diffusion function $g(X_\eta(t))$ in (1.1) is replaced by the vector

$$\vec{g}\left(\vec{X}_\eta(t)\right) = \left(g^1\left(\vec{X}_\eta(t)\right), \ldots, g^d\left(\vec{X}_\eta(t)\right)\right) \in [0,\infty)^d.$$

The dynamics of the d components are *coupled* because the argument of each g^i is the full vector $\vec{X}_\eta(t)$. The analogue of (1.16) reads

$$dZ^i(t) = c_{k+1}\left[\theta^i - Z^i(t)\right]\ dt + \sqrt{(F^{[k]}\vec{g})^i(\vec{Z}(t))}\ dW^i(t), \quad 1 \le i \le d, \quad \vec{Z}(0) = \vec{\theta}. \tag{3.2}$$

The step from one to more dimensions brings about *major* mathematical complications:

- Only for a *limited* class of choices of S and $\vec{g}$ has it been proved that (3.1) has a unique weak solution. A similar problem occurs for (3.2).
- The *stochastic part* of the renormalization program has been carried through *only for special choices* of S and g (on the basis of so-called duality arguments).
- The *analytic part* of the renormalization program is hampered by the fact that it is in general *not* possible to write down an *explicit formula* for the equilibrium distribution of (3.2) and hence for the renormalization transformation F.

We will look at two cases where the *analytic part* of the renormalization program can be completed.

3.1. $S \subset \mathbb{R}^d$ compact convex

Den Hollander and Swart [19] considered the *isotropic* case

$$g^1 = \ldots = g^d = g, \tag{3.3}$$

chose S to be a compact convex subset of $\mathbb{R}^d$, $d \ge 2$, and took for $\mathcal{H}$ the class of functions satisfying:

1. g is locally Lipschitz on S.
2. $g > 0$ on $\mathrm{int}(S)$.
3. $g = 0$ on ∂S.

Define

(a) $\mathcal{H}'$ is the largest subclass of $\mathcal{H}$ for which (1.16) has a unique weak solution and a unique equilibrium.
(b) $\mathcal{H}''$ is the largest subclass of $\mathcal{H}'$ that is closed under F.

Then $\mathcal{H}'$ is the class on which F is *well defined* and $\mathcal{H}''$ is the class on which F can be *iterated*.

In what follows we will *assume* that

$$\mathcal{H}'' = \mathcal{H}' = \mathcal{H}. \tag{3.4}$$

Theorems 3.1–3.2 are taken from [19] and rely on (3.4).

Theorem 3.1. *The solution of the eigenvalue problem $Fg = \lambda g$, $g \in \mathcal{H}$, $\lambda > 0$, is the 1-parameter family*

$$g = dg^* \text{ and } \lambda = \frac{1}{1+d}, \qquad d > 0, \tag{3.5}$$

where g^* is the unique continuous solution of

$$\begin{array}{lcll} \Delta g^* & = & -2 & \text{on } \mathrm{int}(S), \\ g^* & = & 0 & \text{on } \partial S. \end{array} \tag{3.6}$$

Theorem 3.2. *For all $g \in \mathcal{H}$,*

$$\lim_{k\to\infty} kF^k g = g^* \tag{3.7}$$

uniformly on S.

The above g^* takes over the role of the Fisher-Wright diffusion function on the unit interval (recall Section 2.1).

It is believed that the assumption in (3.4) is valid in full generality for the choice of S and $\mathcal{H}$ indicated above, but so far this remains a challenge. Swart [22] has proved that $\mathcal{H}'$ contains all those $g \in \mathcal{H}$ for which there exists an $\epsilon > 0$ such that the level sets $\{x \in S\colon\ g(x) \geq r\}$ are convex for all $0 < r < \epsilon$. The *stochastic part* of the renormalization program is largely open.

Without the isotropy assumption in (3.3), little is known so far. Dawson and March [14] proved that for S the simplex, $\mathcal{H}'$ contains a small neighborhood of the d-dimensional analogue of the Fisher-Wright diffusion. Cerrai and Clément [6], [7] proved that for S the simplex and the hypercube, respectively, $\mathcal{H}'$ contains a large subset of $\mathcal{H}$. This work represents very important progress on the difficult weak uniqueness issue in the anisotropic case. The *stochastic part* of the renormalization program in the anisotropic case is also largely open, with partial progress in Dawson, Greven and Vaillancourt [13] for S the simplex.

3.2. $S = [0,\infty)^2$

In Dawson, Greven, den Hollander, Sun and Swart [12] (work in progress) the two-dimensional version of (3.2) is considered:

$$\begin{aligned} dX^1_\eta(t) &= \sum_{\zeta\in\Omega_N} a_N(\eta,\zeta)\,[X^1_\zeta(t) - X^1_\eta(t)]\,dt + \sqrt{g_1(X^1_\eta(t),X^2_\eta(t))}\,dW^1_\eta(t),\\ dX^2_\eta(t) &= \sum_{\zeta\in\Omega_N} a_N(\eta,\zeta)\,[X^2_\zeta(t) - X^2_\eta(t)]\,dt + \sqrt{g_2(X^1_\eta(t),X^2_\eta(t))}\,dW^2_\eta(t). \end{aligned} \tag{3.8}$$

Here, $\vec{g} = (g_1, g_2)$ is a pair of diffusion functions driving the pair of components $\vec{X}_\eta(t) = (X^1_\eta(t), X^2_\eta(t))$. Unlike in Section 3.1, we will allow the *anisotropic* case $g_1 \neq g_2$. The two-dimensional version of (3.2) (for $c_k \equiv 1$) reads

$$\begin{aligned} dZ^1(t) &= [\theta^1 - Z^1(t)]\,dt + \sqrt{g_1(Z^1(t),Z^2(t))}\,dW^1(t),\\ dZ^2(t) &= [\theta^2 - Z^2(t)]\,dt + \sqrt{g_2(Z^1(t),Z^2(t))}\,dW^2(t). \end{aligned} \tag{3.9}$$

The *stochastic part* of the renormalization program is addressed in Cox, Dawson and Greven [8] for a special case, and is largely open. The *analytic part* is being addressed in [12]. For this part, $\vec{g} = (g_1, g_2)$ is taken from the following class, which we denote by $\mathcal{H}$:

1. $g_1, g_2 > 0$ on $(0,\infty)^2$.
2. $g_1(x_1,x_2) = x_1 h_1(x_1,x_2)$ with either:
   2.a. $h_1 > 0$ on $[0,\infty)^2$ and h_1 Hölder on compact subsets of $[0,\infty)^2$.
   2.b. $h_1(x_1,x_2) = x_2\gamma_1(x_1,x_2)$, $\gamma_1 > 0$ on $[0,\infty)^2$ and γ_1 Hölder on compact subsets of $[0,\infty)^2$.
3. $g_2(x_1,x_2) = x_2 h_2(x_1,x_2)$ with either:
   3.a. $h_2 > 0$ on $[0,\infty)^2$ and h_2 Hölder on compact subsets of $[0,\infty)^2$.
   3.b. $h_2(x_1,x_2) = x_1\gamma_2(x_1,x_2)$, $\gamma_2 > 0$ on $[0,\infty)^2$ and γ_2 Hölder on compact subsets of $[0,\infty)^2$.
4. $g_1(x_1,x_2), g_2(x_1,x_2) \leq C(x_1+1)(x_2+1)$ for some $C = C(g_1,g_2) < \infty$.

Dawson and Perkins [15] (work in progress) show that (3.9) has a unique weak solution under properties 1-3 above. Earlier results in this direction, under stronger restrictions on g, were obtained by Athreya, Barlow, Bass and Perkins [1] and by Bass and Perkins [4], [5]. In [12] it is shown that properties 1-3 are enough to also have a unique equilibrium. Thus, if we define subclasses $\mathcal{H}'' \subset \mathcal{H}' \subset \mathcal{H}$ as in Section 3.1, then we have

$$\mathcal{H}' = \mathcal{H}. \tag{3.10}$$

It is believed that under properties 1–4 above,

$$\mathcal{H}'' = \mathcal{H}, \tag{3.11}$$

although this is still open.

A number of results are derived in [12] subject to (3.11). We cite two results in Theorems 3.3–3.4, subject to the condition that $\vec{g}$ be "sufficiently regular near the boundary and at infinity". The precise regularity conditions are technical. Regularity near the boundary means that g^1 is either everywhere zero or everywhere positive on $\{x_2 = 0\}$, and similarly for g^2 on $\{x_1 = 0\}$. Regularity at infinity means that g^1, g^2 are regularly varying along rays in $[0,\infty)^2$ in a certain uniform sense.

The renormalization transformation F, which acts on the *pair* of diffusion functions $\vec{g} = (g_1, g_2)$, is given by

$$(F\vec{g})(\vec{y}) = \int_{[0,\infty)^2} \vec{g}(\vec{x}) \nu_{\vec{y},\vec{g}}(d\vec{x}), \tag{3.12}$$

where $\nu_{\vec{y},\vec{g}}$ is the equilibrium distribution of (3.9).

Theorem 3.3. *The solution of the eigenvalue problem $F\vec{g} = \lambda\vec{g}$, $\vec{g} \in \mathcal{H}$, $\lambda > 0$, subject to g being "sufficiently regular near the boundary and at infinity", is the 4-parameter family*

$$\vec{g} = \vec{g}_d^* \text{ and } \lambda = Id, \qquad d = (d_1, d_2, d_3, d_4) \geq 0,\ d_1 + d_2 > 0,\ d_3 + d_4 > 0 \tag{3.13}$$

where $\vec{g}_d^* = (g_d^{*,1}, g_d^{*,2})$ has the form

$$\begin{array}{rcl} g_d^{*,1}(x_1, x_2) & = & (d_1 + d_2 x_2) x_1, \\ g_d^{*,2}(x_1, x_2) & = & (d_3 + d_4 x_1) x_2. \end{array} \tag{3.14}$$

Theorem 3.4. *For all $\vec{g} \in \mathcal{H}$ that are "sufficiently regular near the boundary and at infinity",*

$$\frac{k}{e_k} F^k \vec{g} \asymp \vec{g}_d^* \quad \text{as } k \to \infty, \tag{3.15}$$

where d is determined by the behavior of $\vec{g}$ near the boundary and (e_k) by the behavior of $\vec{g}$ at infinity.

For instance, if $g_1 = g_2 = 0$ on $\{x_1 = 0\} \cup \{x_2 = 0\}$ and $g_1(x_1, x_2) = g_2(x_1, x_2) \asymp x_1^\alpha x_2^\beta$ as $x_1, x_2 \to \infty$ with $0 < \alpha \vee \beta \leq 1$, then

$$d = (0, 1, 0, 1) \quad \text{and} \quad e_k \asymp k^{1/(2-\alpha\vee\beta)}. \tag{3.16}$$

The 4 universality classes in Theorem 3.3 correspond to special diffusions:

$d = (>0, 0, >0, 0)$: *non-catalytic* branching.
$d = (0, >0, >0, 0)$: *catalytic* branching.
$d = (>0, 0, 0, >0)$: *catalytic* branching.
$d = (0, >0, 0, >0)$: *mutually catalytic* branching.

These take over the role of the Feller diffusion function on the half-line (recall Section 2.2).

4. Open problems

The main open problems are:

(a) Carry out the stochastic part of the renormalization program in higher dimension (for the examples in, respectively, Sections 3.1 and 3.2).
(b) Prove (3.4) for S the simplex, respectively, the hypercube. Attempt to extend the proof for S an arbitrary compact convex subset of $\mathbb{R}^d$.
(c) Prove (3.11) for $S = [0,\infty)^2$.

It clearly is a challenge to push the renormalization analysis forward to even richer examples.

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It is a pleasure to dedicate this paper to Philippe Clément on the occasion of his retirement. I thank him very warmly for our fruitful and ongoing mathematical collaboration, as well as for the many interesting and enjoyable discussions we have shared over the years.

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Reduced Mihlin-Lizorkin Multiplier Theorem in Vector-valued L^p Spaces

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Dedicated to Professor Philippe Clément on the occasion of his retirement

Abstract. We prove a new Fourier multiplier theorem for operator-valued symbols in UMD spaces with property (α) by making simultaneous use of the various good geometric properties of the Banach spaces in question that are available. Our sufficient condition intersects the known Mihlin-Lizorkin and Hörmander type assumptions.

1. Introduction

It was a long-standing problem to generalize S. G. Mihlin's classical theorem on Fourier-multipliers on $L^p(\mathbf{R}^n)$ to the setting of $\mathscr{L}(X)$-valued multiplier functions acting on the Bôchner spaces $L^p(\mathbf{R}^n, X)$, where X is a Banach space not isomorphic to a Hilbert space. A solution was first obtained by L. Weis [11], whose work exploited decisively the notion of R-boundedness, which had been studied in detail by Ph. Clément, B. de Pagter, F.A. Sukochev and H. Witvliet [3]. Once this breakthrough was achieved, there has been some activity towards obtaining sharper, and even optimal (in a certain sense, cf. [4]), smoothness assumptions for operator-valued multipliers on $L^p(\mathbf{R}^n, X)$ [4, 6, 8, 10]. Perhaps surprisingly, the methods developed in this connection have been able to relax the assumptions from what was known before even for the classical multipliers on $L^p(\mathbf{R}^n)$ [6].

While the sharpest known multiplier conditions are rather technical, the essence of the matter (i.e., the order of required smoothness as a function of the Fourier-type of the underlying Banach spaces) is contained in the following statement: (Recall that every UMD space indeed has some Fourier-type $t \in]1, 2]$.)

1.1. Theorem ([6]). *Let X and Y be UMD spaces with Fourier-type $t \in]1,2]$. Let the function $m \in L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))$ satisfy the R-boundedness condition*

$$M_\alpha := \mathscr{R}[|\xi|^{|\alpha|} D^\alpha m(\xi) : \xi \in \mathbf{R}^n \setminus \{0\}] < \infty \qquad \text{for all} \quad \alpha \in \{0,1\}^n \text{ such that } |\alpha| \leq \lfloor n/t \rfloor + 1.$$

Then m is an $(L^p(\mathbf{R}^n, X), L^p(\mathbf{R}^n, Y))$ Fourier multiplier for $1 < p < \infty$.

Here $\mathscr{R}(\mathscr{T})$ designates the R-bound of a set $\mathscr{T} \subset \mathscr{L}(X,Y)$; see [3, 11] for more on this notion.

The same conclusion, but assuming $M_\alpha < \infty$ for all $\alpha \in \{0,1\}^n$ had been obtained earlier by F. Zimmermann [12] for scalar-valued and by Ž. Štrkalj and L. Weis [10] for operator-multipliers, and assuming $M_\alpha < \infty$ for all $|\alpha| \leq \lfloor n/t \rfloor + 1$ by M. Girardi and Weis [4]. They also show that the smoothness order $\lfloor n/t \rfloor + 1$ cannot be essentially relaxed. (A slight improvement is possible by considering appropriate fractional order smoothness.) The observation from [6] that one actually only needs the intersection of these two different sets of assumptions was new even in $L^p(\mathbf{R}^n)$, where it simultaneously improved the classical multiplier theorems of Mihlin ($\alpha \in \{0,1\}^n$) and L. Hörmander ($|\alpha| \leq \lfloor n/2 \rfloor + 1$).

Whereas the Fourier-type controls the required order of smoothness of multipliers, another geometric notion is known to be related to the order of required decay and admissible blow-up of the derivatives of m. This is the property (α) of G. Pisier, which allows the following form of the multiplier theorem, obtained by Štraklj and Weis:

1.2. Theorem ([10]). *Let X and Y be UMD spaces with property (α). Let $m \in L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))$ satisfy the R-boundedness condition*

$$N_\alpha := \mathscr{R}[\xi^\alpha D^\alpha m(\xi) : \xi \in (\mathbf{R} \setminus \{0\})^n] < \infty \qquad \text{for all} \quad \alpha \in \{0,1\}^n.$$

Then m is an $(L^p(\mathbf{R}^n, X), L^p(\mathbf{R}^n, Y))$ Fourier multiplier for $1 < p < \infty$.

In $L^p(\mathbf{R}^n)$, this improvement of Mihlin's theorem is due to P. I. Lizorkin, and for scalar multipliers on $L^p(\mathbf{R}^n, X)$ due to Zimmermann [12]. It has also been extended to the mixed-norm spaces $L^{\bar{p}}(\mathbf{R}^n, X)$, $\bar{p} = (p_1, \ldots, p_n) \in]1, \infty[^n$ by the author [7]. Zimmermann showed that already for scalar-multipliers, Theorem 1.2 fails to extend to all UMD spaces. Recently Weis and the author observed that its validity actually characterizes UMD spaces with property (α) [9].

If X and Y are UMD spaces with Fourier-type $t \in]1,2]$ and property (α), then one could try to check the boundedness of a multiplier $m \in L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))$ by either Theorem 1.1 or 1.2: the latter one requires in general more derivatives but imposes weaker size conditions on them. A natural question is whether the intersection of these assumptions would suffice. Note that this problem arises already on $L^p(\mathbf{R}^n)$, and an affirmative answer would be an improvement of the classical multiplier theorems by means of intersecting Hörmander's and Lizorkin's conditions.

And indeed we are able to provide such an answer:

1.3. Theorem. *Let X and Y be UMD-spaces with property (α) and Fourier-type $t \in]1,2]$. Let $m \in L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))$ satisfy the R-boundedness condition*

$$N_\alpha = \mathscr{R}[\xi^\alpha D^\alpha m(\xi) : \xi \in (\mathbf{R} \setminus \{0\})^n] < \infty \qquad \text{for all} \quad \alpha \in \{0,1\}^n \text{ such that } |\alpha| \leq \lfloor n/t \rfloor + 1.$$

Then m is an $(L^p(\mathbf{R}^n, X), L^p(\mathbf{R}^n, Y))$ Fourier-multiplier for $1 < p < \infty$.

In fact, Theorem 1.3 will be proved in a slightly more general form than here stated, which also covers some of the mixed-norm spaces $L^{\bar{p}}(\mathbf{R}^n, X)$. There are certain limitations on our techniques in this setting, however, so that we are unable to cover all multi-exponents $\bar{p} \in]1,\infty[^n$, but only a certain subset $\mathscr{P}_t$ thereof. Nevertheless, it is strictly larger than the subset of pure exponents $p \in]1,\infty[$ appearing in the statement of Theorem 1.3 above.

The method of proof relies mainly on ideas of Girardi, Weis and the author from [4, 6, 8], appropriately modified to exploit the additional hypothesis of property (α). In particular, the proof consists of roughly the following two parts:

- obtain a criterion for the boundedness of operator-valued convolutions $f \mapsto k * f$, and then
- use a "Fourier embedding theorem" to check that the multiplier functions m appearing in Theorem 1.3 are mapped by the Fourier transform into the class of convolution kernels k for which the mentioned criterion is verified.

Just like in [6], we find that the "qualitative" UMD and (α) properties are mainly used in the first step, whereas the "quantitative" Fourier-type condition only plays a rôle in the second. A part of Theorem 1.3, namely the multiplier property of m for $t \leq p \leq t'$, can also be obtained by a simpler approach introduced in [7], which essentially consists of an induction on the dimension n, starting from a one-dimensional result from [4]. We indicate this in more detail in the appendix. Note that while the multiplier condition of Theorem 1.1 implies the Hörmander integral condition for the associated kernel k (so that the boundedness of the multiplier on $L^p(\mathbf{R}^n, X)$ for one $p \in]1,\infty[$ already gives the boundedness for all p), the condition of Theorem 1.3 does not, and thus the simpler approach, unfortunately, appears insufficient to recover the full result. The failure of Hörmander's integral condition underlines the more delicate nature of the Lizorkin-type multipliers compared to the Mihlin or Hörmander-type ones.

2. Preliminaries, and a criterion for convolution operators

Let us begin by making some conventions. If $\alpha = (\alpha_i)_{i=1}^n$ and $\beta = (\beta_i)_{i=1}^n$ are two vectors of same length, we define their product componentwise by $\alpha\beta := (\alpha_i\beta_i)_{i=1}^n$. A vector to the power of another vector is defined as the scalar $\alpha^\beta := \prod_{i=1}^n \alpha_i^{\beta_i}$, whereas a scalar t to the power of a vector α is the vector $t^\alpha := (t^{\alpha_i})_{i=1}^n$. A combination which often appears is $2^{-\mu}x$, where $\mu \in \mathbf{Z}^n$ and $x \in \mathbf{R}^n$. By our conventions, this is the vector $(2^{-\mu_i}x_i)_{i=1}^n \in \mathbf{R}^n$. We also record the formula $(t^\alpha)^\beta = t^{\alpha\cdot\beta}$, where $\alpha \cdot \beta := \sum_{i=1}^n \alpha_i\beta_i$ is the usual scalar product.

By ε_μ, $\mu \in \mathbf{Z}^n$, we denote a sequence of independent random variables on some probability space Ω, with distribution $P(\varepsilon_\mu = +1) = P(\varepsilon_\mu = -1)$. These are called Rademacher variables. We denote the mathematical expectation on (Ω, P) by E.

The mixed-norm spaces $L^{\bar{p}}(\mathbf{R}^n, X)$ are defined inductively as follows:

For $n = 1$, $L^{(p_1)}(\mathbf{R}, X) := L^{p_1}(\mathbf{R}, X)$ is the usual Bôchner space.
For $\bar{p} = (\bar{q}, p_n)$, where $\bar{q} = (p_1, \ldots, p_{n-1})$ and $n > 1$, we set
$L^{\bar{p}}(\mathbf{R}^n, X) := L^{p_n}(\mathbf{R}, L^{\bar{q}}(\mathbf{R}^{n-1}, X))$.

This recursive definition is often handy in proving n-dimensional results with a one-dimensional version as a starting point. In this connection it is useful to note that when X is a UMD space, then so is $L^p(\mathbf{R}, X)$ for $p \in]1, \infty[$ and by induction also $L^{\bar{p}}(\mathbf{R}^n, X)$ for $\bar{p} \in]1, \infty[^n$. The same applies to property (α).

We next give a version for the mixed norm spaces of a useful lemma of J. Bourgain [2], which also lies at the heart of the multiplier theorems in [4, 6, 8]. The assumption that X be UMD is used via the validity of the $n = 1$ case of the lemma.

2.1. Lemma. *Let X be a UMD space and $\bar{p} \in]1, \infty[^n$. For a finite set of indices $\mu \in \mathbf{Z}^n$, let*

$$f_\mu \in L^{\bar{p}}(\mathbf{R}^n, X) \quad \textit{with} \quad \hat{f}_\mu \subset 2^\mu \cdot [-1, 1]$$

and

$$h_{(\mu)} = (h^{(1)}_{\mu_1}, \ldots, h^{(n)}_{\mu_n}) \in \mathbf{R}^n \quad \textit{with} \quad |h^{(i)}_j| \leq K_i 2^{-j}, \quad \textit{where} \quad K_1, \ldots, K_n \geq 2.$$

Then

$$E\Big\| \sum_\mu \varepsilon_\mu f_\mu(\cdot - h_{(\mu)})\Big\|_{L^{\bar{p}}(\mathbf{R}^n, X)} \leq C \cdot \prod_{i=1}^n \log K_i \cdot E\Big\| \sum_\mu \varepsilon_\mu f_\mu\Big\|_{L^{\bar{p}}(\mathbf{R}^n, X)}.$$

Proof. The case $n = 1$ is proved in [2] for $\mathbf{T}$ in place of $\mathbf{R}$; this is transferred to $\mathbf{R}$ in [4]. (A generalization to $\mathbf{R}^n$ is also given there, but of a slightly different kind than what we want here.) Let us assume the lemma true for some n, and consider the case $n + 1$. We write $\bar{p} = (\bar{q}, p_{n+1})$. Note that both sides of the estimate to be proved remain invariant if we multiply ε_μ by $\eta_{\mu_{n+1}}$, where $(\eta_j)_{j\in\mathbf{Z}}$ is an independent Rademacher sequence. For $\mu = (\nu, j) \in \mathbf{Z}^n \times \mathbf{Z}$, let us write $h_{(\nu,j)} =: (\tilde{h}_{(\nu)}, h^{(n+1)}_j) \in \mathbf{R}^n \times \mathbf{R}$.

For each $j \in \mathbf{Z}$, consider the function

$$x \in \mathbf{R}^n \mapsto F^\varepsilon_j(x) := \sum_{\nu\in\mathbf{Z}^n} \varepsilon_{(\nu,j)} f_{(\nu,j)}(\cdot - \tilde{h}_{(\nu)}, x) \in L^{\bar{q}}(\mathbf{R}^n, X).$$

Then $\operatorname{supp}\hat{F}_j^\varepsilon \subset 2^j\cdot[-1,1]$, and so we have (applying the $n=1$ case on the UMD Banach space $L^{\bar{q}}(\mathbf{R}^n,X)$)

$$\begin{aligned}
&E\Big\|\sum_\mu \varepsilon_\mu f_\mu(\cdot-h_{(\mu)})\Big\|_{L^{\bar{p}}(\mathbf{R}^{n+1},X)} \leq E\Big\|\sum_j \eta_j F_j^\varepsilon(\cdot-h_j^{(n+1)})\Big\|_{L^{p_{n+1}}(\mathbf{R},L^{\bar{q}}(\mathbf{R}^n,X))}\\
&\leq C\log K_{n+1}\cdot E\Big\|\sum_j \eta_j F_j^\varepsilon\Big\|_{L^{p_{n+1}}(\mathbf{R},L^{\bar{q}}(\mathbf{R}^n,X))} \qquad (2.2)\\
&\leq C\log K_{n+1}\Big(\int_{\mathbf{R}}\Big[E\Big\|\sum_\nu \eta_\nu\sum_j \varepsilon_{(\nu,j)}f_{(\nu,j)}(\cdot-\tilde{h}_{(\nu)},x)\Big\|_{L^{\bar{q}}(\mathbf{R}^n,X)}\Big]^{p_{n+1}}\mathrm{d}x\Big)^{1/p_{n+1}},
\end{aligned}$$

where we used Kahane's inequality and the invariance of the distribution of $\varepsilon_{(\nu,j)}$ under multiplication by an independent Rademacher sequence η_ν.

Observe that the functions

$$G_\nu^{\varepsilon,x} := \sum_j \varepsilon_{(\nu,j)}f_{(\nu,j)}(\cdot,x)\in L^{\bar{q}}(\mathbf{R}^n,X)$$

satisfy $\operatorname{supp}\hat{G}_\nu^{\varepsilon,x}\subset 2^\nu\cdot[-1,1]$, while the vectors $\tilde{h}_{(\nu)}=(h_{\nu_1}^{(1)},\ldots,h_{\nu_n}^{(n)})$ satisfy $|h_j^{(i)}|\leq K_i 2^{-j}$. Thus, by the induction assumption,

$$E\Big\|\sum_\nu \eta_\nu G_\nu^{\varepsilon,x}(\cdot-h_{(\nu)})\Big\|_{L^{\bar{q}}(\mathbf{R}^n,X)}\leq C\cdot\prod_{i=1}^n K_i E\Big\|\sum_\nu \eta_\nu G_\nu^{\varepsilon,x}\Big\|_{L^{\bar{q}}(\mathbf{R}^n,X)}.$$

Applying this estimate in (2.2), and using Kahane's inequality one more time, we arrive at the assertion of the lemma. □

For the next result, we introduce a partition of unity on $\mathbf{R}^n$. Let first $\hat{\varphi}_0^{(1)}\in\mathscr{D}(\mathbf{R})$ be symmetric about the origin, have support in $[-2,2]$, be constantly 1 on $[-1,1]$ and decreasing on $[1,2]$. Let $\hat{\phi}_0^{(1)}(\xi):=\hat{\varphi}_0^{(1)}(\xi)-\hat{\varphi}_0^{(1)}(2\xi)$. We then define a function $\hat{\phi}_0\in\mathscr{D}(\mathbf{R}^n)$ by $\hat{\phi}_0(\xi):=\prod_{i=1}^n\hat{\varphi}_0^{(1)}(\xi_i)$. Finally, for $\mu\in\mathbf{Z}^n$, let $\hat{\phi}_\mu(\xi):=\hat{\phi}_0(2^{-\mu}\xi)$. Then $\sum_{\mu\in\mathbf{Z}^n}\hat{\phi}_\mu(\xi)=1$ for all $\xi\in(\mathbf{R}\setminus\{0\})^n=:\mathbf{R}_*^n$. Naturally, $\phi_\mu(x)=2^{-\mu\cdot\iota}\phi_0(2^{-\mu}x)$, where $\iota:=(1,\ldots,1)$, will be the function whose Fourier transform is $\hat{\phi}_\mu$.

It is useful to introduce

$$\chi_\mu := \sum_{|\nu-\mu|_\infty\leq 1}\phi_\nu,$$

so that $\hat{\chi}_\mu$ is constantly 1 on $\operatorname{supp}\hat{\phi}_\mu$.

The assumption that a certain space X is UMD with property (α) will mainly be used through the following Littlewood-Paley-type theorem on $L^{\bar{p}}(\mathbf{R}^n,X)$, and the similar result on $L^{\bar{p}}(\mathbf{R}^n,X')$, which follows from the known fact that X', too, is a UMD space with property (α) when X is. (Note that unlike the UMD condition, the property (α) alone is not self-dual; nevertheless, the joint property "UMD and (α)" is.)

2.3. Proposition. *Let X be a UMD space with property (α). Then*

$$E\Big\|\sum_\mu \varepsilon_\mu \chi_\mu * f\Big\|_{L^{\bar p}(\mathbf{R}^n,X)} \leq C\,\|f\|_{L^{\bar p}(\mathbf{R}^n,X)}$$

for all $f \in L^{\bar p}(\mathbf{R}^n, X)$ and $\bar p \in]1,\infty[^n$, with C depending only on X and $\bar p$.

For $\bar p = p \cdot \iota$, this follows from the results of Zimmermann [12], while the general case can easily be obtained from [7].

The $n = 1$ case holds in every UMD space, and there are also variants of the Littlewood-Paley decomposition which remain bounded in the multi-dimensional setting for arbitrary UMD spaces. Results analogous to the following have been proved in that setting in [4, 8]. The proof given below is not very different from that in [8], but it is hoped that we have managed to make the argument a little more transparent.

2.4. Proposition. *Let X and Y be UMD spaces with property (α), and let $\bar p \in]1,\infty[^n$. Let the distribution $k \in \mathscr{S}'(\mathbf{R}^n, \mathscr{L}(X,Y))$ satisfy*

$$\begin{aligned}\int_{\mathbf{R}^n} E\Big\|\sum_\mu \varepsilon_\mu 2^{-\mu\cdot\iota}(\phi_\mu * k)(2^{-\mu}y) f_\mu\Big\|_{L^{\bar p}(\mathbf{R}^n,Y)} \prod_{i=1}^n \log(2+|y_i|)\,\mathrm{d}y \\ \leq CE\Big\|\sum_\mu \varepsilon_\mu f_\mu\Big\|_{L^{\bar p}(\mathbf{R}^n,X)}\end{aligned} \tag{2.5}$$

for all finitely non-zero sequences of $f_\mu \in L^{\bar p}(\mathbf{R}^n, X)$. Then $f \mapsto k * f$ is a bounded map from $L^{\bar p}(\mathbf{R}^n, X)$ to $L^{\bar p}(\mathbf{R}^n, Y)$.

Proof. For $f \in X \otimes [\hat{\mathscr{D}}_0(\mathbf{R})]^n$, $g \in Y' \otimes [\hat{\mathscr{D}}_0(\mathbf{R})]^n$, we have

$$\langle g, k * f\rangle = \sum_{\mu\in\mathbf{Z}^n} \langle \chi_\mu * g, (\phi_\mu * k) * (\chi_\mu * f)\rangle = \sum_{\mu\in\mathbf{Z}^n} \Big\langle (\phi_\mu * \tilde k)' * (\chi_\mu * g), \chi_\mu * f\Big\rangle$$

where only finitely many of the $\chi_\mu * g$ and $\chi_\mu * f$ are zero, and $\tilde k$ denotes the reflection of k about the origin. A change of variables gives

$$(\phi_\mu * \tilde k)' * (\chi_\mu * g)(x) = \int_{\mathbf{R}^n} 2^{-\mu\cdot\iota}(\phi_\mu * k)(2^{-\mu}y)'(\chi_\mu * g)(x + 2^{-\mu}y)\,\mathrm{d}y,$$

and then

$$\begin{aligned}&\sum_\mu \Big\langle (\phi_\mu * \tilde k)' * (\chi_\mu * g), \chi_\mu * f\Big\rangle \\ &\quad= \sum_\mu \int_{\mathbf{R}^n} \big\langle 2^{-\mu\cdot\iota}(\phi_\mu * k)(2^{-\mu}y)'(\chi_\mu * g)(\cdot + 2^{-\mu}y), \chi_\mu * f\big\rangle\,\mathrm{d}y \\ &\quad= \int_{\mathbf{R}^n} E\Big\langle \sum_\mu \varepsilon_\mu(\chi_\mu * g)(\cdot + 2^{-\mu}y), \sum_\nu \varepsilon_\nu 2^{-\nu\cdot\iota}(\phi_\nu * k)(2^{-\nu}y)(\chi_\nu * f)\Big\rangle\,\mathrm{d}y.\end{aligned}$$

By Hölder's and Kahane's inequalities, the absolute value of this is bounded by

$$C\int_{\mathbf{R}^n} E\Big\|\sum_\mu \varepsilon_\mu(\chi_\mu * g)(\cdot + 2^{-\mu}y)\Big\|_{L^{\bar{p}'}(\mathbf{R}^n,Y')}$$
$$\times\ E\Big\|\sum_\nu \varepsilon_\nu 2^{-\nu\cdot\iota}(\phi_\nu * k)(2^{-\nu}y)(\chi_\nu * f)\Big\|_{L^{\bar{p}}(\mathbf{R}^n,Y)}\,\mathrm{d}y.$$

By the previous lemma and the Littlewood-Paley theorem on $L^{\bar{p}'}(\mathbf{R}^n,Y')$, we have

$$\begin{aligned} E\Big\|\sum_\nu \varepsilon_\nu(\chi_\nu * g)(\cdot + 2^{-\nu}y)\Big\|_{L^{\bar{p}'}(\mathbf{R}^n,Y')} &\le C\prod_{i=1}^n \log(2+|y_i|)\cdot\Big\|\sum_\nu \varepsilon_\nu\chi_\nu * g\Big\|_{L^{\bar{p}'}(\mathbf{R}^n,Y')} \\ &\le C\prod_{i=1}^n \log(2+|y_i|)\,\|g\|_{L^{\bar{p}'}(\mathbf{R}^n,Y')}\,. \end{aligned}$$

By the assumption and the Littlewood-Paley theorem on $L^{\bar{p}}(\mathbf{R}^n,X)$,

$$\begin{aligned} &\int_{\mathbf{R}^n} E\Big\|\sum_\mu \varepsilon_\mu 2^{-\mu\cdot\iota}(\phi_\mu * k)(2^{-\mu}y)(\chi_\mu * f)\Big\|_{L^{\bar{p}}(\mathbf{R}^n,Y)}\cdot\prod_{i=1}^n \log(2+|y_i|)\,\mathrm{d}y \\ &\le CE\Big\|\sum_\mu \varepsilon_\mu\chi_\mu * f\Big\|_{L^{\bar{p}}(\mathbf{R}^n,X)} \le C\,\|f\|_{L^{\bar{p}}(\mathbf{R}^n,X)}\,. \end{aligned}$$

Everything combined, we have shown that

$$|\langle g, k*f\rangle| \le C\,\|g\|_{L^{\bar{p}'}(\mathbf{R}^n,Y')}\,\|f\|_{L^{\bar{p}}(\mathbf{R}^n,X)}\,,$$

which obviously gives the assertion. □

3. Multiplier theorems via embeddings

The next step on the road to multiplier theorems is to find efficient conditions for checking the assumption of Prop. 2.4 in terms of smoothness and size of the multiplier $m = \hat{k}$. Note that the mentioned assumption is the requirement of membership of a certain function in the logarithmically weighted L^1 space. Thus the following result about an embedding into such a space is not completely unrelated. The proof is similar to that of Lemma 8.1 in [6], but we give the details for the convenience of the reader.

Below, the assumption that X have Fourier-type $t \in\,]1,2]$ enters the scene. Recall that this means the boundedness of the Fourier transform from $L^t(\mathbf{R}^n,X)$ to $L^{t'}(\mathbf{R}^n,X)$.

We also employ the product symbol in connection with multiple integrals in a rather formal way, which is best understood by thinking of "$\prod_{i:\alpha_i=1}\mathscr{E}_i$" simply as "write the expressions $\mathscr{E}_i$, for all i such that $\alpha_i = 1$, in a row, and then interpret what you have in the usual way"; cf. [6].

3.1. Proposition. *Let X be a Banach space with Fourier-type $t \in]1,2]$. Let $w(x) = \prod_{i=1}^n w_i(x_i)$, where each w_i is even, positive and non-decreasing on $\mathbf{R}_+$. Then*

$$\int_{\mathbf{R}^n} |\hat{f}(x)|_X w(x)\,\mathrm{d}x \leq C \sum_{\alpha\in\{0,1\}^n} \prod_{i:\alpha_i=1} \int_0^{1/4} \frac{\mathrm{d}h_i}{h_i} h_i^{-1/t} w_i(h_i^{-1})\, \|\delta_h^\alpha f\|_{L^t(\mathbf{R}^n,X)} \,.$$

The $\alpha = 0$ term on the right means simply $\|f\|_{L^t(\mathbf{R}^n,X)}$.

Proof. We make use of the decomposition of $\mathbf{R}^n$ introduced in [6]. For $\rho \in]0,\infty[^n$, $\alpha \in \{0,1\}^n$ and $j \in \mathbf{N}^\alpha := \{\mu \in \mathbf{N}^n : \mu_i = 0 \text{ if } \alpha_i = 0\}$, let

$$\begin{aligned} E(\alpha,\rho) &:= \{x \in \mathbf{R}^n : |x_i| \leq \rho_i \text{ if } \alpha_i = 0,\ |x_i| > \rho_i \text{ if } \alpha_i = 1\} \\ E(\alpha,\rho,j) &:= \{x \in E(\alpha,\rho) : 2^{j_i}\rho_i < |x_i| \leq 2^{j_i+1}\rho_i \text{ if } \alpha_i = 1\}. \end{aligned}$$

The two key observations are the facts that

$$|1 - e^{\mathbf{i}2\pi x\cdot e_i/2^{j_i+2}\rho_i}| \geq c \qquad \text{for} \qquad x \in E(\alpha,\rho,j),\ \alpha_i = 1,$$

and that $(1-e^{\mathbf{i}2\pi x\cdot h})\hat{f}(x)$ is the Fourier transform of $f - f(\cdot - h) =: f - \tau_h f =: \Delta_h f$ at x. Then

$$\begin{aligned} \int_{E(\alpha,\rho)} |\hat{f}(x)|_X \cdot w(x)\,\mathrm{d}x &= \sum_{j\in\mathbf{N}^\alpha} \int_{E(\alpha,\rho,j)} |\hat{f}(x)|_X \cdot w(x)\,\mathrm{d}x \\ &\leq C \sum_{j\in\mathbf{N}^\alpha} \int_{E(\alpha,\rho,j)} \Big| \prod_{i:\alpha_i=1} (1 - e^{\mathbf{i}2\pi x\cdot e_i/2^{j_i+2}\rho_i}) \cdot \hat{f}(x) \Big|_X w(x)\,\mathrm{d}x \\ &\leq C \sum_{j\in\mathbf{N}^\alpha} \Big\| x \mapsto \prod_{i:\alpha_i=1} (1 - e^{\mathbf{i}2\pi x\cdot e_i/2^{j_i+2}\rho_i}) \cdot \hat{f}(x) \Big\|_{L^{t'}(\mathbf{R}^n,X)} \Big(\int_{E(\alpha,\rho,j)} w^t(x)\,\mathrm{d}x \Big)^{1/t} \\ &\leq \sum_{j\in\mathbf{N}^\alpha} C \Big\| \prod_{i:\alpha_i=1} \Delta_{e_i/2^{j_i+2}\rho_i} f \Big\|_{L^t(\mathbf{R}^n,X)} \prod_{i:\alpha_i=0} w_i(2\rho_i)(2\rho_i)^{1/t} \\ &\qquad \times \prod_{i:\alpha_i=1} w_i(2^{j_i+1}\rho_i)(2^{j_i+1}\rho_i)^{1/t}. \end{aligned}$$

We integrate with respect to $\mathrm{d}\rho_i/\rho_i$ from r to $2r$ for every i:

$$\begin{aligned} &\prod_{i=1}^n \int_r^{2r} \frac{\mathrm{d}\rho_i}{\rho_i} \int_{E(\alpha,\rho)} |\hat{f}(x)|_X \cdot w(x)\,\mathrm{d}x \\ &\quad\leq C r^{(n-|\alpha|)/t} \prod_{i:\alpha_i=0} w_i(4r) \sum_{j\in\mathbf{N}^\alpha} \prod_{i:\alpha_i=1} \int_r^{2r} \frac{\mathrm{d}\rho_i}{\rho_i} w_i(2^{j_i+1}\rho_i)(2^{j_i+1}\rho_i)^{1/t} \\ &\qquad \times \Big\| \prod_{i:\alpha_i=1} \Delta_{e_i/2^{j_i+2}\rho_i} f \Big\|_{L^t_X} \end{aligned}$$

$$
\begin{aligned}
&= Cr^{(n-|\alpha|)/t} \prod_{i:\alpha_i=0} w_i(4r) \sum_{j\in\mathbf{N}^\alpha} \prod_{i:\alpha_i=1} \int_{1/2^{j_i+3}r}^{1/2^{j_i+2}r} \frac{\mathrm{d}h_i}{h_i} w_i(1/2h_i)(2h_i)^{-1/t} \\
&\quad\times \left\|\delta_h^\alpha f\right\|_{L^t(\mathbf{R}^n,X)} \\
&\leq Cr^{(n-|\alpha|)/t} \prod_{i:\alpha_i=0} w_i(4r) \times \prod_{i:\alpha_i=1} \int_0^{1/4r} \frac{\mathrm{d}h_i}{h_i} h_i^{-1/t} w_i(h_i^{-1}) \left\|\delta_h^\alpha f\right\|_{L^t(\mathbf{R}^n,X)},
\end{aligned}
$$

where we have adopted the notation $\delta_h^\alpha := \prod_{i:\alpha_i=1} \Delta_{h_i e_i}$.

Summing over all $\alpha \in \{0,1\}^n$ and taking $r = 1$, we get the assertion. □

Again, we introduce some more notation. Since in the interpolation of $L^{\bar{p}}$ spaces a key rôle is played by the fact that the reciprocal of a certain multi-exponent can be expressed as a linear combination of others, we define

$$
\begin{aligned}
\operatorname{conv}_{-1} \mathscr{A} :=& (\operatorname{conv} \mathscr{A}^{-1})^{-1} \\
=& \{\bar{p} : 1/\bar{p} = \sum_{j=1}^N \sigma_j/\bar{p}^{(j)},\ \bar{p}^{(j)} \in \mathscr{A},\ \sigma_j \geq 0,\ \sum_{j=1}^N \sigma_j = 1\}
\end{aligned}
$$

Let us denote $\iota_k := (1,\dots,1) \in \mathbf{R}^k$, and define

$$
\mathscr{A}_t := \operatorname{conv}_{-1} \bigcup_{k=0}^n [t,t']^k \times \{\iota_{n-k}\}.
$$

We also recall that $\operatorname{Rad} X$ is the completion of all finitely non-zero sums $\sum_\mu \varepsilon_\mu x_\mu$ in $L^2(\Omega, X)$. By Kahane's inequality it follows that we could equally well define this as a completion in $L^r(\Omega, X)$ for any $r \in [1,\infty[$, and then by Fubini's theorem $\operatorname{Rad}(L^p(\mathbf{R},X)) \approx L^p(\mathbf{R}, \operatorname{Rad} X)$, and by induction $\operatorname{Rad}(L^{\bar{p}}(\mathbf{R}^n,X)) \approx L^{\bar{p}}(\mathbf{R}^n, \operatorname{Rad} X)$ for all $p \in [1,\infty[$, resp. $\bar{p} \in [1,\infty[^n$. See, e.g., [4] for more on this space and its use in the present kind of connection.

3.2. Lemma. *Let Y have Fourier-type $t \in]1,2]$, and let $k \in \mathscr{S}'(\mathbf{R}^n, \mathscr{L}(X,Y))$ with $m := \hat{k} \in L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))$. Let*

$$
\begin{aligned}
M := \sum_{\alpha\in\{0,1\}^n} \prod_{i:\alpha_i=1} \int_0^{1/4} \frac{\mathrm{d}h_i}{h_i} h_i^{-1/t} \log(h_i^{-1}) \\
\times \left\| \mathscr{R}[\delta_h^\alpha(\hat{\phi}_0(\cdot) m(2^\mu \cdot)) : \mu \in \mathbf{Z}^n | \mathscr{L}(X,Y)] \right\|_t
\end{aligned}
\tag{3.3}
$$

be finite. Then (2.5) *holds with $C = c(\bar{p},X)M < \infty$ for all $f_\mu \in L^{\bar{p}}(\mathbf{R}^n, X)$ and $\bar{p} \in \mathscr{A}_t$.*

Proof. Let first $f_\mu \in L^{\bar{q}}(\mathbf{R}^k, X)$, where $\bar{q} \in [t,t']^k$ and $k \in \{0,1,\dots,n\}$; for $k = 0$, we understand this simply as $f_\mu \in X$. For a fixed y, we have $2^{\mu\cdot\iota}(\phi_\mu * k)(2^{-\mu}y) \in \mathscr{L}(X,Y)$, and this operator has a canonical extension to $\mathscr{L}(L^{\bar{q}}(\mathbf{R}^k,X), L^{\bar{q}}(\mathbf{R}^k,Y)$. For $\bar{q}$ in the described range, the space $L^{\bar{q}}(\mathbf{R}^k, Y)$ has Fourier-type t, and the same is still true of the space $\operatorname{Rad}(L^{\bar{q}}(\mathbf{R}^k, Y)) \approx L^{\bar{q}}(\mathbf{R}^k, \operatorname{Rad} Y)$. It follows from Prop. 3.1 that

$$\begin{aligned}
&\int_{\mathbf{R}^n} \Big\| \sum_\mu \varepsilon_\mu 2^{-\mu\cdot\iota}(\phi_\mu * k)(2^{-\mu}y) f_\mu \Big\|_{L^{\bar{q}}(\mathbf{R}^k, \operatorname{Rad} Y)} \prod_{i=1}^n \log(2+|y_i|)\, \mathrm{d}y \\
&\leq C \sum_{\alpha\in\{0,1\}^n} \prod_{i:\alpha_i=1} \int_0^{1/4} \frac{\mathrm{d}h_i}{h_i} h_i^{-1/t} \log(2+h_i^{-1}) \\
&\qquad \times \Big\| \delta_h^\alpha \sum_\mu \varepsilon_\mu (\hat{\phi}_\mu \hat{k})(2^\mu \cdot) f_\mu \Big\|_{L^t(\mathbf{R}^n, L^{\bar{q}}(\mathbf{R}^k, \operatorname{Rad} Y))} \\
&\leq C \sum_{\alpha\in\{0,1\}^n} \prod_{i:\alpha_i=1} \int_0^{1/4} \frac{\mathrm{d}h_i}{h_i} h_i^{-1/t} \log(h_i^{-1}) \\
&\qquad \times \Big\| \mathscr{R}\{\delta_h^\alpha(\hat{\phi}_0(\cdot) m(2^\mu \cdot))\}_{\mu\in\mathbf{Z}^n} \Big\|_t \Big\| \sum_\mu \varepsilon_\mu f_\mu \Big\|_{L^{\bar{q}}(\mathbf{R}^k, \operatorname{Rad} X)} \\
&= CM \Big\| \sum_\mu \varepsilon_\mu f_\mu \Big\|_{L^{\bar{q}}(\mathbf{R}^k, \operatorname{Rad} X)}.
\end{aligned}$$

Next, consider a multi-exponent $\bar{p} = (\bar{q}, \iota_{n-k})$, where $\bar{q}$ is as above. Let $F_\mu \in L^{\bar{p}}(\mathbf{R}^n, X) = L^{\iota_{n-k}}(\mathbf{R}^{n-k}, L^{\bar{q}}(\mathbf{R}^k, X))$. Then the partial point-evaluations $F_\mu(\cdot, x)$, $x \in \mathbf{R}^{n-k}$, belong to $L^{\bar{q}}(\mathbf{R}^k, X)$, so the estimate just established applies to $f_\mu = F_\mu(\cdot, x)$. If we integrate the resulting inequality with respect to $\mathrm{d}x$ on $\mathbf{R}^{n-k}$, we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^n} \Big\| \sum_\mu \varepsilon_\mu 2^{-\mu\cdot\iota}(\phi_\mu * k)(2^{-\mu}y) F_\mu \Big\|_{L^{\bar{p}}(\mathbf{R}^n, \operatorname{Rad} Y)} \\
&\qquad \times \prod_{i=1}^n \log(2+|y_i|)\, \mathrm{d}y \leq CM \Big\| \sum_\mu \varepsilon_\mu F_\mu \Big\|_{L^{\bar{p}}(\mathbf{R}^n, \operatorname{Rad} X)}.
\end{aligned}$$

Thus we have established the desired boundedness

$$L^{\bar{p}}(\mathbf{R}^n, \operatorname{Rad} X) \to L^1(\mathbf{R}^n, \prod_{i=1}^n \log(2+|y_i|)\, \mathrm{d}y; L^{\bar{p}}(\mathbf{R}^n, \operatorname{Rad} Y))$$

for all $\bar{p} \in \bigcup_{k=0}^n [t, t']^k \times \{\iota_{n-k}\}$. It suffices to apply, say, the complex interpolation method (more precisely, see [1], 5.1.2) to extend this to all the exponents in the assertion. □

Now we are ready for a multiplier theorem. The version given below is the most general of this paper, and just like the most general versions in [4, 6] has rather technical assumptions. Hopefully more attractive versions are given as Corollaries below, where it is shown, in particular, that Theorem 1.3 follows from this abstract version.

3.4. Theorem. *Let X and Y be UMD spaces with property (α), each of Fourier-type $t \in]1,2]$. Let $m \in L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))$ be such that the quantity M in (3.3) is finite. Let us denote*

$$\mathscr{B}_t := \mathscr{A}_t \cap]1,\infty[^n, \qquad \mathscr{B}'_t := \{\bar{p}' : \bar{p} \in \mathscr{B}_t\}, \qquad \mathscr{P}_t := \mathrm{conv}_{-1}(\mathscr{B}_t \cup \mathscr{B}'_t).$$

Then m is an $(L^{\bar{p}}(\mathbf{R}^n, X), L^{\bar{p}}(\mathbf{R}^n, Y))$ Fourier-multiplier for all $\bar{p} \in \mathscr{P}_t$, in particular for all $\bar{p} = p \cdot \iota$, $p \in]1,\infty[$.

Proof. The assertion for $\bar{p} \in \mathscr{B}_t$ is an immediate consequence of Prop. 2.4 and Lemma 3.2. On the other hand, the assumptions of the Proposition remain invariant on replacing X by Y', Y by X' and m by $\xi \mapsto m(\xi)' \in \mathscr{L}(Y', X')$. Hence it also follows that $m(\cdot)'$ is an $(L^{\bar{p}}(\mathbf{R}^n, Y'), L^{\bar{p}}(\mathbf{R}^n, X'))$ Fourier-multiplier for $\bar{p} \in \mathscr{B}_t$, and this implies by duality that m is an $(L^{\bar{p}}(\mathbf{R}^n, X), L^{\bar{p}}(\mathbf{R}^n, Y))$ for $\bar{p} \in \mathscr{B}'_t$. The full assertion now follows by interpolation. □

3.5. Corollary. *Let X and Y be UMD spaces with property (α) and Fourier-type $t \in]1,2]$. Let $t^{-1} < \gamma \leq 1$, and let $m \in L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))$ satisfy the following R-boundedness condition:*

$$L_\alpha := \mathscr{R}[\xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta_\eta^\alpha m(\xi) : \alpha \in \{0,1\}^n,\ |\xi_i| > 2\,|\eta_i| > 0] < \infty$$

for all $\alpha \in \{0,1\}^n$. Then m is an $(L^{\bar{p}}(\mathbf{R}^n, X), L^{\bar{p}}(\mathbf{R}^n, Y))$ Fourier-multiplier for all $\bar{p} \in \mathscr{P}_t$.

The case $n = 1$, which holds even without property (α), is due to Girardi and Weis [4].

Proof. Obviously, it suffices to show that the quantity M in (3.3) is bounded by the sum of L_α's. It is readily verified that

$$\delta_h^\alpha(f \cdot g) = \sum_{\theta \leq \alpha} \tau_{\theta h}\delta_h^{\alpha-\theta} f \cdot \delta_h^\theta g.$$

Thus, for $h \in]0, 1/4]^n$,

$$\mathscr{R}[\delta_h^\alpha(\hat{\phi}_0(\xi)m(2^\mu\xi)) : \mu \in \mathbf{Z}^n] \leq \sum_{\theta \leq \alpha} |\tau_{\theta h}\delta_h^{\alpha-\theta}\hat{\phi}_0(\xi)| \cdot \mathscr{R}[(\delta_{2^\mu h}^\theta m)(2^\mu\xi) : \mu \in \mathbf{Z}^n].$$

Note that the condition that $\tau_{\theta h}\delta_h^{\alpha-\theta}\hat{\phi}_0(\xi) \neq 0$ forces $|\xi_i| \approx 1$ for all $i = 1, \ldots, n$. The R-bound above is dominated by $L_\theta h^{\theta\gamma}$. On the other hand, since $\hat{\phi}_0$ is a smooth function, we have $|\delta_h^{\alpha-\theta}\hat{\phi}_0(\xi)| \leq Ch^{\alpha-\theta} \leq Ch^{(\alpha-\theta)\gamma}$. Taking moreover into account the finite support of $\hat{\phi}_0$, we find that

$$\left\| \mathscr{R}[\delta_h^\alpha(\hat{\phi}_0(\cdot)m(2^\mu\cdot)) : \mu \in \mathbf{Z}^n] \right\|_t \leq Ch^{\alpha\gamma} \sum_{\theta \leq \alpha} L_\theta,$$

and then

$$M \leq C \sum_{\alpha \in \{0,1\}^n} \prod_{i:\alpha_i = 1} \int_0^{1/4} \frac{\mathrm{d}h_i}{h_i} h_i^{-1/t+\gamma} \log(h_i^{-1}) \times \sum_{\theta \leq \alpha} L_\theta,$$

and all the integrals are convergent, since $\gamma > 1/t$. □

The specialization of the following Corollary to the case of a pure exponent $\bar{p} = p \cdot \iota$, $p \in]1, \infty[$, is Theorem 1.3, which was stated in the introduction.

3.6. Corollary. *Let X and Y be a UMD space with property (α) and Fourier-type $t \in]1, 2]$. If $m \in L^\infty(\mathbf{R}^n, \mathscr{L}(X, Y))$ satisfies*

$$\mathscr{R}[\xi^\alpha D^\alpha m(\xi) : \xi \in \mathbf{R}^n_*,\ \alpha \in \{0,1\}^n,\ |\alpha| \leq \lfloor n/t \rfloor + 1] < \infty,$$

then m is an $(L^{\bar{p}}(\mathbf{R}^n, X), L^{\bar{p}}(\mathbf{R}^n, Y))$ Fourier multiplier for all $\bar{p} \in \mathscr{P}_t$.

Proof. It suffices to pick $1/t < \gamma \leq 1$ and show that the R-bounds assumed in the previous Corollary can be estimated by the one assumed now. Note that $n/t < \lfloor n/t \rfloor + 1 \leq n$, so that we may choose γ in such a way that $n/t < n\gamma \leq \lfloor n/t \rfloor + 1$.

Let $\alpha \in \{0,1\}^n$. If $|\alpha| \leq \lfloor n/t \rfloor + 1$, then we obtain at once that

$$\begin{aligned}
\xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta^\alpha_\eta m(\xi) &= \xi^{\alpha\gamma}\eta^{-\alpha\gamma}\int_{[0,1]^\alpha} \eta^\alpha D^\alpha m(\xi - u\eta)\,\mathrm{d}u \\
&= \eta^{\alpha(1-\gamma)}\xi^{\alpha(\gamma-1)}\int_{[0,1]^\alpha} \frac{\xi^\alpha}{(\xi - u\eta)^\alpha}(\xi - u\eta)^\alpha D^\alpha m(\xi - u\eta)\,\mathrm{d}u,
\end{aligned}$$

so that

$$\mathscr{R}[\xi^{\alpha\gamma}\eta^{-\alpha\gamma} : |\xi_i| > 2\,|\eta_i| > 0] \leq 2^{|\alpha|\gamma}\mathscr{R}[\xi^\alpha D^\alpha m(\xi) : \xi \in \mathbf{R}^n_*].$$

If $\alpha > \lfloor n/t \rfloor + 1$, then a somewhat more careful argument is required. Consider a splitting $\alpha = \beta + \theta$; $\beta, \theta \geq 0$, where $|\beta| = \lfloor n/t \rfloor + 1 > n/t$, and then $|\theta| < n/t'$. For each of the finite number of such splittings, we consider

$$\mathscr{R}[\xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta^\alpha_\eta m(\xi) : 2 < |\xi_i/\eta_i| \leq |\xi_j/\eta_j| \ \text{ for } \beta_j = 1,\ \theta_i = 1].$$

Note that the sets $\{2 < |\xi_i/\eta_i| \leq |\xi_j/\eta_j| \ \text{ for } \beta_j = 1,\ \theta_i = 1\}$, when (β, θ) ranges over all splittings of α as above, cover the set $\{|\xi_i| > 2\,|\eta_i| > 0\}$, for among the $|\alpha|$ indices i with $\alpha_i = 1$, there always exists a collection of $\lfloor n/t \rfloor + 1$ indices j for which the ratio $|\xi_j/\eta_j|$ is at least as large as for the remaining indices.

Then we have

$$\begin{aligned}
\xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta^\alpha_\eta m(\xi) &= \xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta^\theta_\eta \int_{[0,1]^\beta} \eta^\beta D^\beta m(\xi - u\eta)\,\mathrm{d}u \\
&= \xi^{\theta\gamma}\eta^{-\theta\gamma}\xi^{\beta(\gamma-1)}\eta^{\beta(1-\gamma)}\sum_{\kappa \leq \theta}(-1)^{|\kappa|} \\
&\quad \times \int_{[0,1]^\beta} \frac{\xi^\beta}{(\xi - (u+\kappa)\eta)^\beta}(\xi - (u+\kappa)\eta)^\beta D^\beta m(\xi - (u+\kappa)\eta)\,\mathrm{d}u,
\end{aligned}$$

so that

$$\begin{aligned}
&\mathscr{R}[\xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta^\alpha_\eta m(\xi) : 2 < |\xi_i/\eta_i| \leq |\xi_j/\eta_j| \ \text{ for } \beta_j = 1,\ \theta_i = 1] \\
&\quad \leq \sup\Big[\big|\xi^{\theta\gamma}\eta^{-\theta\gamma}\big| \cdot |\xi^{\beta(\gamma-1)}\eta^{\beta(1-\gamma)}| : 2 < |\xi_i/\eta_i| \leq |\xi_j/\eta_j| \ \text{ for } \beta_j = 1,\ \theta_i = 1\Big] \\
&\qquad\qquad \times 2^{|\beta|}\mathscr{R}[\xi^\beta D^\beta m(\xi) : \xi \in \mathbf{R}^n_*].
\end{aligned}$$

In the set under consideration, we have

$$\begin{aligned}|\xi^{\theta\gamma}\eta^{-\theta\gamma}|\cdot|\xi^{\beta(\gamma-1)}\eta^{\beta(1-\gamma)}| &\leq |\xi^{\beta\gamma}\eta^{-\beta\gamma}|^{|\theta|/|\beta|}\cdot|\xi^{\beta(\gamma-1)}\eta^{\beta(1-\gamma)}|\\ &= |\xi^{\beta}\eta^{-\beta}|^{\gamma(|\theta|/|\beta|+1)-1}.\end{aligned}$$

Since $\left|\xi^{\beta}\eta^{-\beta}\right| > 2^{|\beta|}$, it remains to show that $\gamma(|\theta|/|\beta|+1)-1\leq 0$. But

$$\gamma(|\theta|+|\beta|)=\gamma\,|\alpha|\leq\gamma n\leq\lfloor n/t\rfloor+1=|\beta|\,,$$

which is equivalent to this claim. □

One more corollary is in order: it says that if a family of multipliers satisfies the assumptions of the previous Corollaries in a uniform way, not only are the individual operators bounded, but in fact the whole family is again R-bounded. The proof is omitted, since it is an immediate consequence of the previous results by using the general bootstrapping method for operator-valued multiplier theorems on spaces with property (α), which was invented by Girardi and Weis in [5].

3.7. Corollary. *Let X and Y be UMD spaces with property (α) and Fourier-type $t\in\,]1,2]$. Let $\mathscr{M}\subset L^{\infty}(\mathbf{R}^n,\mathscr{L}(X,Y))$ be a collection of multipliers which satisfies*

$$\mathscr{R}[\xi^{\alpha}D^{\alpha}m(\xi):\alpha\in\{0,1\}^n,\ |\alpha|\leq\lfloor n/t\rfloor+1,\ m\in\mathscr{M}]<\infty,$$

or more generally, with $1/t<\gamma\leq 1$,

$$\mathscr{R}[\xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta_{\eta}^{\alpha}m(\xi):\alpha\in\{0,1\}^n,\ |\xi_i|>2\,|\eta_i|>0,\ m\in\mathscr{M}]<\infty.$$

Then the collection $\mathscr{T}$ of all $(L^{\bar{p}}(\mathbf{R}^n,X),L^{\bar{p}}(\mathbf{R}^n,Y))$ Fourier-multipliers associated with $m\in\mathscr{M}$ is an R-bounded set for all $\bar{p}\in\mathscr{P}_t$, and $\mathscr{R}(\mathscr{T})$ is estimated in terms of the R-bounds in the assumptions.

4. Appendix: Another approach to a special case of the main theorem

In [7] it was shown that certain multiplier theorems in UMD spaces with property (α) admit a relatively simple proof by induction on the dimension n, starting from the one-dimensional result. In particular, a new proof of Theorem 1.2 was given by this method. It is natural to ask whether this inductive approach could also yield the improved Theorem 1.3.

The answer is in part "yes", but with certain limitations. This approach has the problem that while $L^p(\mathbf{R},X)$ (and then $L^{\bar{p}}(\mathbf{R}^n,X)$) inherits the UMD and (α) properties of X for all $p\in\,]1,\infty[$ (resp. $\bar{p}\in\,]1,\infty[^n$), this is only true for the Fourier-type t property for $p\in[t,t']$ (resp. $\bar{p}\in[t,t']^n$). This restricts the multi-exponents we are able to cover with such approach.

But to see what can be done, we give the following proposition, whose statement is just a specialization of Cor. 3.7 but for which we can give a simpler proof. (Of course, simplicity is a conditional property depending on our prior knowledge, and in the present case our "simpler proof" relies essentially on the $n=1$ case from [4], whose proof is not in any substantial way easier than that of the general result above.)

4.1. Proposition. *Let X and Y be UMD spaces with property (α) and Fourier-type $t \in]1,2]$. For an $(L^{\bar{p}}(\mathbf{R}^n, X), L^{\bar{p}}(\mathbf{R}^n, Y))$ Fourier-multiplier m, denote by T_m the corresponding operator between these spaces. Then for $\mathscr{M} \subset L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))$,*

$$\mathscr{R}[T_m : m \in \mathscr{M}] \leq C\mathscr{R}[\xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta_\eta^\alpha m(\xi) : \alpha \in \{0,1\}^n,\ |\xi_i| > 2\,|\eta_i| > 0,\ m \in \mathscr{M}]$$

for all $\bar{p} \in]1,\infty[\times[t,t']^{n-1}$ and $1/t < \gamma \leq 1$, where C depends only on X, $\bar{p}$ and γ.

Proof. For $n = 1$ and $\mathscr{M} = \{m\}$, this was proved by Girardi and Weis [4] (without property (α)), and for a general $\mathscr{M}$ (and assuming (α)) this follows from their bootstrapping method [5].

If we assume the theorem valid for some n, we can use the induction method from [7] to prove it for $n+1$. In fact, the $(n+1)$-dimensional multiplier operator T_m, with $m \in L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))$ acting on $L^{(\bar{q},p)}(\mathbf{R}^{n+1}, X) = L^p(\mathbf{R}, L^{\bar{q}}(\mathbf{R}^n, X))$ is naturally identified (at least on a suitable test function class) with the one-dimensional multiplier operator $T_{\tilde{m}}$, where $\tilde{m}(x) := T_{m(\cdot,x)}$. Thus

$$\begin{aligned}
&\mathscr{R}[T_m : m \in \mathscr{M} | \mathscr{L}(L^{(\bar{q},p)}(\mathbf{R}^{n+1}, X), L^{(\bar{q},p)}(\mathbf{R}^{n+1}, Y))] \\
&= \mathscr{R}[T_{\tilde{m}} : m \in \mathscr{M} | \mathscr{L}(L^p(\mathbf{R}, L^{\bar{q}}(\mathbf{R}^n, X)), L^p(\mathbf{R}, L^{\bar{q}}(\mathbf{R}^n, Y)))] \\
&\leq C\mathscr{R}[x^{\alpha\gamma}y^{-\alpha\gamma}\delta_y^\alpha \tilde{m}(x) : \alpha \in \{0,1\},\ |x| > 2\,|y| > 0,\ m \in \mathscr{M} | \\
&\qquad\qquad\qquad\qquad\qquad\qquad \mathscr{L}(L^{\bar{q}}(\mathbf{R}^n, X), L^{\bar{q}}(\mathbf{R}^n, Y))] \\
&= C\mathscr{R}[T_{x^{\alpha\gamma}y^{-\alpha\gamma}\delta^\alpha_{ye_{n+1}}m(\cdot,x)} : \alpha \in \{0,1\},\ |x| > 2\,|y| > 0,\ m \in \mathscr{M}] \\
&\leq C\mathscr{R}[\xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta_\eta^\alpha m(\xi) : \alpha \in \{0,1\}^{n+1},\ |\xi_i| > 2\,|\eta_i| > 0, m \in \mathscr{M} | \mathscr{L}(X,Y)],
\end{aligned}$$

where the first inequality was an application of Girardi and Weis' $n = 1$ case on the spaces $L^{\bar{q}}(\mathbf{R}^n, X)$ and $L^{\bar{q}}(\mathbf{R}^n, Y)$, both of which are UMD with (α) and Fourier-type t, and the second used the induction assumption. □

By repeating the reasoning in the proof of Cor. 3.6, one sees that for $\bar{p} \in]1,\infty[\times[t,t']^{n-1}$, the mentioned Corollary also follows from the previous Proposition, and in particular we get Theorem 1.3 for $p \in [t,t']$, as we claimed.

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Interpolation Spaces for Initial Values of Abstract Fractional Differential Equations

Stig-Olof Londen

Dedicated to Philippe Clément on the occasion of his retirement

Abstract. We consider the parabolic evolutionary equation

$$D_t^\beta(u_t - z) + Au = f, \quad u(0) = y, \quad u_t(0) = z,$$

in continuous interpolation spaces allowing a singularity as $t \to 0$. Here $\beta \in (0,1)$. We analyze the corresponding trace spaces and show how they depend on β and on the strength of the singularity. The results are applied to the quasilinear equation

$$D_t^\beta(u_t - z) + A(u, u_t)u = f, \quad u(0) = y, \quad u_t(0) = z.$$

1. Introduction

In a recent paper [3] we analyzed quasilinear parabolic evolutionary equations of type

$$D_t^\alpha(u - x) + A(u)u = f(u) + h(t), \quad t > 0, \quad u(0) = x, \tag{1.1}$$

in continuous interpolation spaces allowing a singularity in u as $t \downarrow 0$. Here D_t^α denotes the time-derivative of order $\alpha \in (0,2)$.

First, we gave a treatment of fractional derivatives in the spaces $L^p((0,T);X)$, X a Banach space, and next considered these derivatives in spaces of X-valued continuous functions having (at most) a prescribed singularity as $t \downarrow 0$. The appropriate trace spaces were characterized, including their dependence on α. Via maximal regularity results on the linear equation

$$D_t^\alpha(u - x) + Au = f(t), \quad t > 0, \quad u(0) = x,$$

we then obtained statements on quasilinear equations of type (1.1).

In the case where $\alpha \in (1,2)$ and the parameters are such that u_t is pointwise well defined, our approach only dealt with the case where $u_t(0) = 0$. This is clearly a drawback, in particular in view of the applications.

The case where $u_t(0)$ is nonzero does however require some additional nontrivial analysis. Statements concerning this case cannot simply be deduced from those of [3].

In this paper, we carry through the required additional analysis. Our main results are given in Section 2. In particular, in Theorem 2.2, we characterize the appropriate trace spaces involved. Proposition 2.3 recalls an earlier result on the necessary maximal regularity.

Taking $u(0)$, $u_t(0)$ as given by Theorem 2.2 we formulate, in Theorem 2.4 a result concerning the solution u of the linear equation

$$D_t^\beta(u_t - z) + Au = f, \quad u(0) = y, \quad u_t(0) = z.$$

Here $\beta = \alpha - 1$. The spaces where u, u_t are continuous on $[0, T]$ are given and we make explicit the norm estimates needed for the quasilinear case.

Our statements on the quasilinear equation

$$D_t^\beta(u_t - z) + A(u, u_t)u = f(u, u_t) + h(t), \quad t > 0, \quad u(0) = y, \quad u_t(0) = z, \tag{1.2}$$

are given in Theorem 2.5. Here we allow A to depend not only on u as in (1.1) , but also on u_t. At the end of Section 2 we briefly comment on an application of Theorem 2.5.

The proofs of Theorem 2.2 and Theorem 2.4 are given in Section 3 and Section 4, respectively. Once Theorem 2.2 and Theorem 2.4 are available, then Theorem 2.5 is proved in much the same way as the corresponding result in [3]. Therefore, we do not repeat it here.

For more references to the literature, and for further connections to earlier work, we refer the reader to [3].

2. Main results

Let E_1, E_0 be Banach spaces, with $E_1 \subset E_0$ and dense embedding, and let A be an isomorphism mapping E_1 into E_0. Throughout we take $\alpha \in (1, 2)$, $\mu \in (0, 1)$ (this is the parameter characterizing the singularity), and write $\beta = \alpha - 1$. Further, let the linear operator A, as an operator in E_0, be nonnegative with spectral angle ϕ_A satisfying

$$2\phi_A < \pi(1 - \beta). \tag{2.1}$$

We assume

$$\mu + \beta > 1, \tag{2.2}$$

and write $J = [0, T]$. We employ the notation

$$E_\theta \stackrel{\text{def}}{=} (E_0, E_1)_\theta \stackrel{\text{def}}{=} (E_0, E_1)^0_{\theta,\infty}, \quad \theta \in (0, 1),$$

for the continuous interpolation spaces between E_0 and E_1. Recall that if η is some number such that

$$0 \le \eta < \pi - \phi_A,$$

then

$$x \in E_\theta \qquad \text{iff} \lim_{|\lambda|\to\infty, |\arg\lambda| \le \eta} \|\lambda^\theta A(\lambda I + A)^{-1}x\|_{E_0} = 0,$$

and that we may take

$$\|x\|_\theta \stackrel{\text{def}}{=} \sup_{|\arg\lambda|\le\eta} \|\lambda^\theta A(\lambda I + A)^{-1}x\|_{E_0},$$

as norm on E_θ.

We recall the definition of a fractional derivative.

Let X be a Banach space and write

$$g_\beta(t) = \frac{1}{\Gamma(\beta)} t^{\beta-1}, \quad t > 0, \quad \beta > 0.$$

Definition 2.1. *Let $u \in L^1((0,T);X)$ for some $T > 0$. We say that u has a fractional derivative of order $\alpha > 0$ provided $u = g_\alpha * f \stackrel{\text{def}}{=} \int_0^t g_\alpha(t-s)f(s)\,ds$, for some $f \in L^1((0,T);X)$. If this holds, write $D_t^\alpha u = f$.*

Next, we specify the singularities involved. Define, for X a Banach space,

$$\begin{aligned} BUC_{1-\mu}(J;X) = \big\{\, u \in C((0,T];X) \,\big|\, & t^{1-\mu}u(t) \in BUC((0,T];X), \\ & \lim_{t\to 0} t^{1-\mu}\|u(t)\|_X = 0 \,\big\}. \end{aligned}$$

We consider the spaces

$$\tilde{E}_0(J) \stackrel{\text{def}}{=} BUC_{1-\mu}(J;E_0), \tag{2.3}$$

$$\tilde{E}_1(J) \stackrel{\text{def}}{=} BUC_{1-\mu}(J;E_1) \cap BUC^1_{1-\mu}(J;E_{\frac{\beta}{1+\beta}}) \cap BUC^{1+\beta}_{1-\mu}(J;E_0), \tag{2.4}$$

where

$$BUC^1_{1-\mu}(J;E_{\frac{\beta}{1+\beta}}) = \big\{\, u \in C^1((0,T];E_{\frac{\beta}{1+\beta}}) \,\big|\, u, u' \in BUC_{1-\mu}(J;E_{\frac{\beta}{1+\beta}}) \,\big\},$$

and

$$\begin{aligned} BUC^{1+\beta}_{1-\mu}(J;E_0) = \big\{\, u \in BUC^1_{1-\mu}(J;E_0) \,\big|\, & u' = z + g_\beta * f, \\ & \text{for some} \quad z \in E_0, \quad f \in \tilde{E}_0(J) \,\big\}. \end{aligned}$$

Thus $u \in BUC^{1+\beta}_{1-\mu}(J;E_0)$ means that $u, u' \in \tilde{E}_0(J)$, that $u'(0) = z$, and that $u' - z$ has a fractional derivative f of order $\beta \in (0,1)$, such that $f \in \tilde{E}_0(J)$. (Cf. [3, Sect. 3].) Observe that by (2.2), and with $f \in \tilde{E}_0(J)$, one has that $g_\beta * f$ is pointwise well defined in E_0 on $[0,T]$; and equals zero for $t = 0$. (Cf. [3, p. 423].)

We equip $\tilde{E}_0(J)$, $\tilde{E}_1(J)$ with the respective norms

$$\|u\|_{\tilde{E}_0(J)} \stackrel{\text{def}}{=} \sup_{0<t\le T} t^{1-\mu}\|u(t)\|_{E_0},$$

$$\|u\|_{\tilde{E}_1(J)} \stackrel{\text{def}}{=} \sup_{0<t\le T} t^{1-\mu}\Big(\|u(t)\|_{E_1} + \|u'(t)\|_{E_{\frac{\beta}{1+\beta}}} + \|f(t)\|_{E_0}\Big),$$

where f is defined through the fact that $u \in \tilde{E}_1(J)$ implies $u = y + tz + g_{1+\beta} * f$, for some $y, z \in E_0$ and $f \in \tilde{E}_0(J)$.

Without loss of generality, take $\|y\|_{E_1} = \|Ay\|_{E_0}$, and note that $\tilde{E}_0(J)$, $\tilde{E}_1(J)$ are Banach spaces (cf. [3, p. 423, Lemma 3]).

Our first purpose is to investigate the trace space of $\tilde{E}_1(J)$.

We define $\gamma : \tilde{E}_1(J) \to (E_0 \times E_0)$ by $\gamma(u) = (u(0), u'(0))$, and the trace space $\gamma(\tilde{E}_1(J)) \stackrel{\text{def}}{=} Im(\gamma)$, with

$$\|(x,y)\|_{\gamma(\tilde{E}_1(J))} \stackrel{\text{def}}{=} \inf\{\, \|v\|_{\tilde{E}_1(J)} \bigm| \gamma(v) = (x,y) \,\}.$$

Define (recall (2.2)),

$$\hat{\mu} = \frac{\mu + \beta}{1 + \beta}, \qquad \tilde{\mu} = \frac{\mu + \beta - 1}{1 + \beta}. \tag{2.5}$$

Then

$$0 < \tilde{\mu} < \mu < \hat{\mu} < 1.$$

Concerning $\gamma(\tilde{E}_1(J))$ we have the following result.

Theorem 2.2. *Take $\mu, \beta \in (0,1)$ such that $\mu + \beta > 1$. Define $\tilde{\mu}, \hat{\mu}$ as in (2.5). Let E_0, E_1 be Banach spaces, with E_1 densely imbedded in E_0. Assume there exists an isomorphism A mapping E_1 into E_0 such that A, as an operator in E_0, is nonnegative, satisfies (2.1), and is such that*

$$D_t^\beta u_t + Au = f, \quad u(0) = u_t(0) = 0, \tag{2.6}$$

has maximal regularity in $\tilde{E}_0(J)$.

Then $\gamma(\tilde{E}_1(J)) = E_{\hat{\mu}} \times E_{\tilde{\mu}}$.

In Section 3 we prove this result through a series of Lemmas. Here we only make some preparatory remarks.

A comparison of Theorem 2.2 with [3, Theorem 7, p. 428] reveals that the price paid for an inclusion of u_t in the characterization of the trace space is the assumption on maximal regularity of (2.6).

As to the question of when this maximal regularity holds, we recall the following. (See [3, Theorem 11, p. 435].)

Let F_1, F_0 be Banach spaces such that

$$E_1 \subset F_1 \subset E_0 \subset F_0, \tag{2.7}$$

and assume that there is an isomorphism $\tilde{A} : F_1 \to F_0$ such that $\tilde{A}$, as an operator in F_0, is nonnegative with spectral angle $\phi_{\tilde{A}}$ satisfying

$$2\phi_{\tilde{A}} < \pi(1 - \beta), \tag{2.8}$$

and such that for some $\theta \in (0,1)$,

$$E_0 = F_\theta \stackrel{\text{def}}{=} (F_0, F_1)_\theta^{0,\infty}, \tag{2.9}$$

and such that

$$Ax = \tilde{A}x, \qquad x \in E_1. \tag{2.10}$$

We then have

Proposition 2.3. *Let the assumptions of Theorem* 2.2*, except the assumption on maximal regularity of* (2.6)*, hold. Assume* (2.7)*,* (2.9) *and suppose there exists an isomorphism* $\tilde{A} : F_1 \to F_0$ *satisfying* (2.8)*,* (2.10)*. Then* (2.6) *has maximal regularity in* $\tilde{E}_0(J)$*.*

Let A be as in Theorem 2.2. Consider the equation (where $y \in E_0$)

$$D_t^\beta v_t + Av = 0, \quad t \in J, \quad v(0) = y, \quad v_t(0) = 0, \tag{2.11}$$

or, equivalently,

$$v - y + g_{1+\beta} * Av = 0, \quad t \in J. \tag{2.12}$$

The formal solution $v(t)$ is given by the (absolutely convergent) integral

$$v(t) = (2\pi i)^{-1} \int_{\Gamma_{1,\psi}} e^{\lambda t} \lambda^\beta \big(\lambda^{1+\beta} I + A\big)^{-1} y \, d\lambda, \quad t > 0, \tag{2.13}$$

where $\psi \in (\frac{\pi}{2}, \frac{\pi - \phi_A}{1+\beta})$, and

$$\Gamma_{r,\psi} \stackrel{\text{def}}{=} \big\{ re^{it} \bigm| |t| \le \psi \big\} \cup \big\{ \rho e^{i\psi} \bigm| r < \rho < \infty \big\} \cup \big\{ \rho e^{-i\psi} \bigm| r < \rho < \infty \big\}.$$

The function defined in (2.13) does in fact satisfy $D_t^\beta v_t + Av = 0$ for $t > 0$. It is a strict solution (i.e., Av, $D_t^\beta v_t \in C([0,T]; E_0)$) of (2.11) provided in addition $y \in \mathcal{D}(A)$. See [2, Lemma 3(g), p. 241]. Moreover, note that because $\mathcal{D}(A)$ is dense in E_0, one has

$$\|v(t) - y\|_{E_0} \to 0, \quad t \downarrow 0, \quad if \quad y \in E_0 \setminus \mathcal{D}(A).$$

In a similar vein, consider (with $z \in E_0$),

$$D_t^\beta (w_t - z) + Aw = 0, \quad t \in J, \quad w(0) = 0, \ w_t(0) = z, \tag{2.14}$$

or, equivalently,

$$w - tz + g_{1+\beta} * Aw = 0, \quad t \in J. \tag{2.15}$$

The formal solution of this equation is

$$w(t) = (2\pi i)^{-1} \int_{\Gamma_{1,\psi}} e^{\lambda t} \lambda^{-1+\beta} \big(\lambda^{1+\beta} I + A\big)^{-1} z \, d\lambda, \quad t > 0, \tag{2.16}$$

with ψ, $\Gamma_{1,\psi}$ as above. The function w so defined does in fact satisfy $D_t^\beta (w_t - z) + Aw = 0$, $t > 0$. It is a strict solution of (2.14) iff $z \in E_{\frac{\beta}{1+\beta}}$. See [2, Lemma 5, p. 242]. Moreover, $w(t) \to 0$ in E_0 for $t \downarrow 0$ and, again by the density of the domain of A in E_0, one has $\|w_t - z\|_{E_0} \to 0$, for $t \downarrow 0$, $z \in E_0 \setminus E_{\frac{\beta}{1+\beta}}$.

Thus, the solution u of

$$D_t^\beta (u_t - z) + Au = 0, \quad t > 0, \quad u(0) = y, \quad u_t(0) = z, \tag{2.17}$$

for $y, z \in E_0$, is formally given by $u = v + w$, where v, w, are given, respectively, by (2.13) and (2.14). This function u satisfies $\lim_{t\downarrow 0} u(t) = y$ and $\lim_{t\downarrow 0} A^{-\delta}u_t = A^{-\delta}z$, for $\delta > (1+\beta)^{-1}$ in E_0; this in case we only know that $y, z \in E_0$.

From Theorem 2.2, and after some calculations, one deduces the following result, needed for the proof of Theorem 2.5.

Theorem 2.4. *Let A, E_0, E_1 be as in Theorem 2.2, including the assumption on maximal regularity of (2.6) in $\tilde{E}_0(J)$. Take μ, β, $\tilde{\mu}$, $\hat{\mu}$ as in Theorem 2.2. Suppose $y \in E_{\hat{\mu}}$, $z \in E_{\tilde{\mu}}$. Let $f \in \tilde{E}_0(J)$ and let u be the solution of*

$$D_t^{\beta}(u_t - z) + Au = f, \quad u(0) = y, \quad u_t(0) = z.$$

Then

$$u \in C([0,T]; E_{\hat{\mu}}), \quad u_t \in C([0,T]; E_{\tilde{\mu}}).$$

Moreover, for some C, independent of u, y, z,

$$\sup_{0\le t\le T} \|u(t)\|_{E_{\hat{\mu}}} + \sup_{0\le t\le T} \|u_t(t)\|_{E_{\tilde{\mu}}} \le C\big(\|y\|_{E_{\hat{\mu}}} + \|z\|_{E_{\tilde{\mu}}} + \|f\|_{\tilde{E}_0(J)}\big).$$

Our last statement concerns the quasilinear equation (1.2).

Take

$$\mu, \beta \in (0,1); \quad \mu + \beta > 1. \tag{2.18}$$

Define $\tilde{\mu}, \hat{\mu}$ as earlier. For X, Y Banach spaces, and g a mapping of X into Y, write $g \in C^{1-}(X;Y)$ if every point $x \in X$ has a neighborhood U such that g restricted to U is globally Lipschitz continuous.

We need to define sets of operators of maximal regularity.

Suppose E_0, E_1 are Banach spaces such that $E_1 \subset E_0$ with dense imbedding. Let $H_\beta(E_1, E_0, \omega)$, $\omega \ge 0$, denote the set of linear continuous mappings $A \in L(E_1, E_0)$ such that $A_\omega \overset{\text{def}}{=} \omega I + A$ is nonnegative, closed in E_0 with spectral angle $< \pi(\frac{1}{2} - \frac{\beta}{2})$. Define

$$H_\beta(E_1, E_0) \overset{\text{def}}{=} \bigcup_{\omega\ge 0} H_\beta(E_1, E_0, \omega),$$

and

$$M_{\beta\mu}(E_1, E_0) \overset{\text{def}}{=} \{\, A \in H_\beta(E_1, E_0) \mid$$
$$D_t^{\beta} u_t + Au = f, \quad u(0) = u_t(0) = 0, \text{ has maximal regularity in } \tilde{E}_0(J)\,\}.$$

In (1.2), now assume

$$(A, f) \in C^{1-}\big(E_{\hat{\mu}} \times E_{\tilde{\mu}}; M_{\beta\mu}(E_1, E_0) \times E_0\big), \tag{2.19}$$

and that

$$y \in E_{\hat{\mu}}, \quad z \in E_{\tilde{\mu}}, \quad h \in BUC_{1-\mu}([0,T]; E_0), \tag{2.20}$$

for any $T > 0$.

We define a solution u of (1.2) on an interval $J = [0,T]$ as a function u satisfying u, $u_t \in C(J; E_0)$, $u \in C((0,T]; E_1)$, $u(0) = y$, $u_t(0) = z$, and such that the fractional derivative of $u_t - z$ of order β satisfies $D_t^{\beta}(u_t - z) \in C((0,T]; E_0)$ and such that (1.2) holds on $0 < t \le T$.

Our result is

Theorem 2.5. *Let* (2.18)–(2.20) *hold. Then there exists a unique maximal solution* u *of* (1.2)*, defined on the maximal interval of existence* $[0,\tau(y,z))$*, where* $\tau(y,z)\in(0,\infty]$*, and such that for every* $T<\tau(y,z)$ *one has*

(i) $u\in BUC_{1-\mu}([0,T];E_1)\cap BUC([0,T];E_{\hat\mu})$,

(ii) $u_t\in BUC^{\beta}_{1-\mu}([0,T];E_0)\cap BUC([0,T];E_{\tilde\mu})$,

(iii) $u+g_\alpha * A(u,u_t)u = y+tz+g_\alpha*(f(u,u_t)+h), \quad 0\le t\le T.$

If $\tau(y,z)<\infty$*, then either* $u\notin UC([0,\tau(y,z);E_{\hat\mu})$ *or* $u'\notin UC([0,\tau(y,z));E_{\tilde\mu})$*.*

If $\tau(y,z)<\infty$*, and* $E_1\subset\subset E_0$*, then for any* $\epsilon>0$*,*

$$\limsup_{t\uparrow\tau(y,z)}\left(\|u(t)\|_{E_{\hat\mu+\epsilon}}+\|u_t(t)\|_{E_{\tilde\mu+\epsilon}}\right)=\infty.$$

Theorem 2.5 can be applied to the quasilinear equation

$$u=y+tz+g_{1+\beta}*(\sigma(u_x)_x),\quad t>0,\quad x\in(0,1), \tag{2.21}$$

with $u=u(t,x)$, and $u(t,0)=u(t,1)=0$, $t\ge0$; $y(x)=u(0,x)$, $z(x)=u_t(0,x)$.

Take σ smooth, with $\sigma(0)=0$, and suppose $0<\sigma_0\le\sigma'(y)\le\sigma_1$, $y\in R$, for some constants σ_0, σ_1.

Suppose we fix $\beta=\frac12$; then $\mu\in(\frac12,1)$ and $\hat\mu=\frac{2\mu+1}{3}$, $\tilde\mu=\frac{2\mu-1}{3}$. Choose $\theta\in(0,\frac12)$, and let

$$E_0=\{\,u\mid u\in h^{2\theta}[0,1];\ u(0)=u(1)=0\,\},$$
$$E_1=\{\,u\in C^2[0,1]\mid u''\in E_0;\ u^{(i)}(0)=u^{(i)}(1)=0, i=0,2\,\}.$$

(Here h denotes the small Hölder spaces.) For specificity, fix $\mu=\frac34$. Then one has

Corollary 2.6. *Assume* $y\in h^{2\theta+\frac53}[0,1]$*,* $z\in h^{2\theta+\frac13}[0,1]$*, with Dirichlet boundary conditions. Then* (2.21)*, with* $\beta=\frac12$*, has a unique maximal solution defined on the maximal interval of existence* $[0,\tau(y,z))$*, such that for any* $T<\tau(y,z)$*,*

$$u\in BUC_{\frac14}([0,T];h^{2\theta+2}[0,1])\cap BUC([0,T];h^{2\theta+\frac53}[0,1]),$$
$$u_t\in BUC^{\frac12}_{\frac14}([0,T];h^{2\theta}[0,1])\cap BUC([0,T];h^{2\theta+\frac13}[0,1]).$$

If τ *is finite, then for any* $\delta>0$*,*

$$\limsup_{t\uparrow\tau}\left(\|u(t)\|_{C^{2\theta+\frac53+\delta}}+\|u_t(t)\|_{C^{2\theta+\frac13+\delta}}\right)=\infty.$$

3. Proof of Theorem 2.2

We show first that if $\varphi\in\tilde E_1(J)$, then $\varphi(0)\in E_{\hat\mu}$, $\varphi_t(0)\in E_{\tilde\mu}$. (Recall that by the assumption $\mu+\beta>1$, both $\varphi(0)$ and $\varphi'(0)$ are well defined in E_0). Thus, let

$$\varphi\in\tilde E_1(J),\quad y\overset{\text{def}}{=}\varphi(0),\quad z\overset{\text{def}}{=}\varphi_t(0). \tag{3.1}$$

Then, by the definition of $\tilde E_1(J)$,

$$D^\beta_t(\varphi_t-z)+A\varphi=h,\quad\text{for some}\quad h\in\tilde E_0(J). \tag{3.2}$$

Let ϑ be defined through

$$D_t^\beta(\vartheta_t) + A\vartheta = h, \quad \vartheta(0) = \vartheta_t(0) = 0. \tag{3.3}$$

Note that by (2.6) one has

$$D_t^\beta(\vartheta_t),\ A\vartheta \in \tilde{E}_0(J). \tag{3.4}$$

We show the following on ϑ_t.

Lemma 3.1. *Let $h \in \tilde{E}_0(J)$ and suppose ϑ solves* (3.3). *Fix $\epsilon \in (0, 1-\mu)$. (Thus $\epsilon \in (0, \beta)$.) Then*

$$\vartheta_t \in BUC_{1-\mu-\epsilon}(J; E_{\frac{\beta-\epsilon}{1+\beta}}). \tag{3.5}$$

Proof. Fix $t > 0$. Then, after some analysis,

$$\vartheta_t(t) = (2\pi i)^{-1} \int_{\Gamma_{1,\psi}} e^{\lambda t}\lambda(\lambda^{1+\beta}I + A)^{-1}\tilde{g}(\lambda)\, d\lambda, \tag{3.6}$$

where $\tilde{g}(\lambda) = \int_0^t e^{-\lambda\tau}h(\tau)\, d\tau$. Observe that by the fact that $\mu + \beta > 1$, one has that the above integral over $\Gamma_{1,\psi}$ converges absolutely.

The claim (3.5) holds provided we show that for any $\tilde{\epsilon} > 0$ there exists $T_{\tilde{\epsilon}} > 0$ such that

$$\sup_{\eta > 0} \left\| \eta^{\frac{\beta-\epsilon}{1+\beta}} A(\eta I + A)^{-1}\vartheta_t(t) \right\|_{E_0} \le \tilde{\epsilon} t^{\mu+\epsilon-1}, \quad 0 < t \le T_{\tilde{\epsilon}}.$$

To this end we write, using (3.6), analyticity, and a change of variables,

$$\begin{aligned}
&\eta^{\frac{\beta-\epsilon}{1+\beta}} A(\eta I + A)^{-1}\vartheta_t(t) \\
&= (2\pi i)^{-1} \int_{\Gamma_{\frac{1}{t},\psi}} \eta^{\frac{\beta-\epsilon}{1+\beta}} A(\eta I + A)^{-1} e^{\lambda t}\lambda(\lambda^{1+\beta}I + A)^{-1}\tilde{g}(\lambda)\, d\lambda \\
&= (2\pi i)^{-1} \int_{\Gamma_{1,\psi}} \eta^{\frac{\beta-\epsilon}{1+\beta}} A^{\frac{1+\epsilon}{1+\beta}}(\eta I + A)^{-1} e^{s}(st^{-1}) A^{\frac{\beta-\epsilon}{1+\beta}}((st^{-1})^{1+\beta}I + A)^{-1}\tilde{g}(st^{-1})t^{-1}\, ds.
\end{aligned}$$

Observe that

$$\sup_{\eta > 0} \left\| \eta^{\frac{\beta-\epsilon}{1+\beta}} A^{\frac{1+\epsilon}{1+\beta}}(\eta I + A)^{-1} \right\|_{L(E_0,E_0)} < \infty,$$

and that, analogously,

$$\sup_{s \in \Gamma_{1,\psi}} \left\| (st^{-1})^{1+\epsilon} A^{\frac{\beta-\epsilon}{1+\beta}} \left((st^{-1})^{1+\beta}I + A\right)^{-1} \right\|_{L(E_0,E_0)} < \infty,$$

with a bound independent of $t > 0$. Thus

$$\|\vartheta_t(t)\|_{E_{\frac{\beta-\epsilon}{1+\beta}}} \le ct^{-1+\epsilon} \int_{\Gamma_{1,\psi}} \|e^s s^{-\epsilon}\tilde{g}(st^{-1})\|_{E_0}\, ds.$$

Estimating, for $s \in \Gamma_{1,\psi}$ and using the assumption $h \in \tilde{E}_0(J)$, gives

$$\begin{aligned}\|e^s \tilde{g}(st^{-1})\|_{E_0} &\le \int_0^t \|e^{s-\frac{s\tau}{t}} h(\tau)\|_{E_0}\, d\tau \\ &\le \tilde{\epsilon} \int_0^t |e^{s-\frac{s\tau}{t}} \tau^{\mu-1}|\, d\tau \le \tilde{\epsilon} \int_0^t e^{-\gamma|s|[1-\frac{\tau}{t}]} \tau^{\mu-1}\, d\tau \\ &= \tilde{\epsilon} t^{\mu} \int_0^1 e^{-\gamma|s|y} (1-y)^{\mu-1}\, dy \le c\tilde{\epsilon} t^{\mu} |s|^{-1},\end{aligned}$$

for any $\tilde{\epsilon} > 0$, some $\gamma, T_{\tilde{\epsilon}} > 0$; $t \in (0, T_{\tilde{\epsilon}}]$, and c independent of $\tilde{\epsilon}$. Hence

$$\|e^s s^{-\epsilon} \tilde{g}(st^{-1})\|_{E_0} \le c\tilde{\epsilon} t^{\mu} |s|^{-1-\epsilon}, \quad t \in (0, T_{\tilde{\epsilon}}],$$

and so

$$\|\vartheta_t(t)\|_{E_{\frac{\beta-\epsilon}{1+\beta}}} \le c\tilde{\epsilon} t^{\epsilon+\mu-1},$$

for any $\tilde{\epsilon} > 0$, $0 < t \le T_{\tilde{\epsilon}}$, and c depending on ϵ and μ but independent of $\tilde{\epsilon}$. So (3.5) holds and Lemma 3.1 is proved. □

Having proved Lemma 3.1, we note that a combination of (3.4), (3.5) yields

$$\vartheta \in \tilde{E}_{1,\epsilon}(J), \tag{3.7}$$

where

$$\tilde{E}_{1,\epsilon}(J) \stackrel{\text{def}}{=} BUC_{1-\mu}(J; E_1) \cap BUC^1_{1-\mu-\epsilon}(J; E_{\frac{\beta-\epsilon}{1+\beta}}) \cap BUC^{1+\beta}_{1-\mu}(J; E_0). \tag{3.8}$$

Let u be defined by $u = \varphi - \vartheta$. Then u satisfies

$$D_t^{\beta}(u_t - z) + Au = 0, \quad u(0) = y, \quad u_t(0) = z, \tag{3.9}$$

and we have that u can be written as the sum of two functions in, respectively, $\tilde{E}_1(J)$ and $\tilde{E}_{1,\epsilon}(J)$. For this case we have the following result.

Lemma 3.2. *Let $y, z \in E_0$, let u be the corresponding solution of* (3.9)*, and assume that $u = u_1 + u_2$, where for some $\epsilon \in (0, 1-\mu)$,*

$$u_1 \in \tilde{E}_1(J), \quad u_2 \in \tilde{E}_{1,\epsilon}(J). \tag{3.10}$$

Then

$$y \in E_{\hat{\mu}}, \quad z \in E_{\tilde{\mu}}. \tag{3.11}$$

Proof. First note that by the representations (2.13), (2.16), and using $y, z \in E_0$, one easily has that for any $T > 0$ there exists $c = c(T, \beta, \|y\|_{E_0}, \|z\|_{E_0})$ such that

$$\|u_t\|_{E_0} + \|Au\|_{E_0} \le c, \quad T \le t < \infty. \tag{3.12}$$

By (3.10), (3.12) it follows that the Laplace transforms of u_t, Au are well defined (in E_0) as absolutely convergent integrals for $\lambda > 0$. Simple calculations give, for $\lambda > 0$,

$$z = \lambda \int_0^{\infty} e^{-\lambda t} u'(t)\, dt + \lambda^{1-\beta} \int_0^{\infty} e^{-\lambda t} Au(t)\, dt. \tag{3.13}$$

Then, for $\eta > 0$,

$$\eta^{\tilde{\mu}} A(\eta I + A)^{-1} z = I_1 + I_2, \tag{3.14}$$

where

$$I_1 \stackrel{\text{def}}{=} \lambda \int_0^\infty e^{-\lambda t} \eta^{\tilde{\mu}} A(\eta I + A)^{-1} u'(t)\, dt,$$

$$I_2 \stackrel{\text{def}}{=} \lambda^{1-\beta} \int_0^\infty e^{-\lambda t} \eta^{\tilde{\mu}} A(\eta I + A)^{-1} Au(t)\, dt.$$

First consider I_1. Take $T_{\hat{\epsilon}} > 0$ (to be fixed below) and write

$$\begin{aligned} I_1 = \lambda \int_0^{T_{\hat{\epsilon}}} e^{-\lambda t} \eta^{\tilde{\mu}} A(\eta I + A)^{-1} u_1'(t)\, dt + \\ \lambda \int_0^{T_{\hat{\epsilon}}} e^{-\lambda t} \eta^{\tilde{\mu}} A(\eta I + A)^{-1} u_2'(t)\, dt + \lambda \int_{T_{\hat{\epsilon}}}^\infty e^{-\lambda t} \eta^{\tilde{\mu}} A(\eta I + A)^{-1} u'(t)\, dt. \end{aligned} \tag{3.15}$$

Consider the first integral on the right side of (3.15). Take any $\hat{\epsilon} > 0$. Then, if $T_{\hat{\epsilon}} > 0$ is sufficiently small, by the first part of (3.10), for $\eta > 0$,

$$\begin{aligned} &\left\|\lambda \int_0^{T_{\hat{\epsilon}}} e^{-\lambda t} t^{\mu-1} \eta^{\tilde{\mu} - \frac{\beta}{1+\beta}} t^{1-\mu} \eta^{\frac{\beta}{1+\beta}} A(\eta I + A)^{-1} u_1'(t)\, dt\right\|_{E_0} \\ &\le \hat{\epsilon} \lambda \int_0^{T_{\hat{\epsilon}}} e^{-\lambda t} t^{\mu-1} \eta^{\tilde{\mu} - \frac{\beta}{1+\beta}}\, dt. \end{aligned} \tag{3.16}$$

Note that (3.13) is valid for any $\lambda > 0$. Therefore, we may choose λ so that $\lambda^{1+\beta} = \eta$. Use this on the right side of (3.16), and extend the integration to $[0, \infty)$ to see that this right side can be estimated by $\hat{\epsilon}\Gamma(\mu)$.

Consider then the second integral on the right side of (3.15). For any $\hat{\epsilon} > 0$, if $T_{\hat{\epsilon}} > 0$ sufficiently small, and by the fact that $u_2' \in BUC_{1-\mu-\epsilon}(J; E_{\frac{\beta-\epsilon}{1+\beta}})$,

$$\begin{aligned} &\left\|\lambda \int_0^{T_{\hat{\epsilon}}} e^{-\lambda t} \eta^{\tilde{\mu} - \frac{\beta-\epsilon}{1+\beta}} \eta^{\frac{\beta-\epsilon}{1+\beta}} A(\eta I + A)^{-1} u_2'(t)\, dt\right\|_{E_0} \\ &\le \hat{\epsilon} \lambda \eta^{\frac{\mu+\epsilon-1}{1+\beta}} \int_0^{T_{\hat{\epsilon}}} e^{-\lambda t} t^{\mu+\epsilon-1}\, dt \le \hat{\epsilon}\Gamma(\epsilon + \mu), \end{aligned}$$

where again we let $\eta = \lambda^{1+\beta}$ to obtain the last inequality.

There remains the estimation of the last integral on the right side of (3.15). Use (3.12) to get (again, obviously $\eta = \lambda^{1+\beta}$),

$$\left\|\lambda \int_{T_{\hat{\epsilon}}}^\infty e^{-\lambda t} \eta^{\tilde{\mu}} A(\eta I + A)^{-1} u'(t)\, dt\right\|_{E_0} \le c(T_{\hat{\epsilon}}) \int_{T_{\hat{\epsilon}}}^\infty \lambda e^{-\lambda t} \eta^{\tilde{\mu}}\, dt.$$

But with $T_{\hat{\epsilon}}$ fixed, this last expression tends to zero as $\lambda \to \infty$, or equivalently, as $\eta \to \infty$. Thus

$$\|I_1(\eta)\|_{E_0} \to 0, \quad as \quad \eta \to \infty. \tag{3.17}$$

Finally, consider I_2 and split the integral in parts, over $[0, T_{\hat\epsilon}]$ and $[T_{\hat\epsilon}, \infty)$, respectively. By the fact that $t^{1-\mu}\|Au(t)\|_{E_0} \to 0$, $t \downarrow 0$, and because

$$\sup_{\eta>0}\|A(\eta I + A)^{-1}\|_{L(E_0,E_0)} < \infty, \tag{3.18}$$

there follows, for any $\hat\epsilon > 0$, provided $T_{\hat\epsilon} > 0$ is sufficiently small and with $\eta = \lambda^{1+\beta}$,

$$\begin{aligned}&\big\|\lambda^{1-\beta}\int_0^{T_{\hat\epsilon}} e^{-\lambda t}\eta^{\tilde\mu}A(\eta I + A)^{-1}Au(t)\,dt\big\|_{E_0}\\ &\le \hat\epsilon\lambda^{\mu}\int_0^{T_{\hat\epsilon}} e^{-\lambda t}t^{\mu-1}\,dt < \hat\epsilon\int_0^{\infty} e^{-s}s^{\mu-1}\,ds = \hat\epsilon\Gamma(\mu).\end{aligned}$$

For the part of I_2 over $[T_{\hat\epsilon}, \infty)$ one has, by (3.12), (3.18),

$$\big\|\lambda^{1-\beta}\int_{T_{\hat\epsilon}}^{\infty} e^{-\lambda t}\eta^{\tilde\mu}A(\eta I + A)^{-1}Au(t)\,dt\big\|_{E_0} \le c\lambda^{1-\beta}\int_{T_{\hat\epsilon}}^{\infty} e^{-\lambda t}\lambda^{(1+\beta)\tilde\mu}\,dt. \tag{3.19}$$

For $\lambda \to \infty$ ($\eta \to \infty$), the right side of (3.19) tends to zero. We conclude that $\lim_{\eta\to\infty}\|I_2(\eta)\|_{E_0} = 0$. By this fact and by (3.14), (3.17),

$$z \in E_{\tilde\mu}. \tag{3.20}$$

To continue the proof of Lemma 3.2, our next goal is to show that the part of u generated by z, denoted w, satisfies,

$$t^{1-\mu}\|Aw(t)\|_{E_0} \to 0, \quad t^{1-\mu}\|w_t(t)\|_{E_{\frac{\beta}{1+\beta}}} \to 0, \tag{3.21}$$

both as $t \downarrow 0$. Thus, assume (3.20) and let w satisfy

$$D_t^{\beta}(w_t - z) + Aw = 0, \quad t \in J, \quad w(0) = 0, \quad w_t(0) = z. \tag{3.22}$$

Equivalently, let w be given by the right side of (2.16). We claim that then (3.21) holds.

Take (2.16), use analyticity to change the integration path to $\Gamma_{\frac{1}{t},\psi}$ and perform a change of variables, to get, by the assumption on z, for any $\hat\epsilon > 0$ if $T_{\hat\epsilon}$ sufficiently small,

$$\|t^{1-\mu}Aw(t)\|_{E_0} \le \hat\epsilon\int_{\Gamma_{1,\psi}} |e^s s^{-\mu}|\,|ds| = c\hat\epsilon, \quad 0 < t \le T_{\hat\epsilon}.$$

To obtain the second part of (3.21), write

$$\begin{aligned}&t^{1-\mu}\eta^{\frac{\beta}{1+\beta}}A(\eta I + A)^{-1}w_t\\ &= (2\pi i)^{-1}t^{-\mu}\int_{\Gamma_{1,\psi}} e^s(st^{-1})^{\beta}\eta^{\frac{\beta}{1+\beta}}A^{\frac{1}{1+\beta}}(\eta I + A)^{-1}A((st^{-1})^{1+\beta}I + A)^{-1}A^{-\frac{1}{1+\beta}}z\,ds.\end{aligned}$$

But $z \in E_{\tilde\mu}$ implies $A^{-\frac{1}{1+\beta}}z \in E_{\frac{\mu+\beta}{1+\beta}}$, see [1, Theorem 10, p. 2247], and so

$$\lim_{t\to 0}\big\|(st^{-1})^{\mu+\beta}A((st^{-1})^{1+\beta}I + A)^{-1}A^{-\frac{1}{1+\beta}}z\big\|_{E_0} = 0,$$

uniformly for $s \in \Gamma_{1,\psi}$. One also has, for some constant c_β,

$$\|A^{\frac{1}{1+\beta}}(\eta I + A)^{-1}\|_{L(E_0,E_0)} \le c_\beta |\eta|^{-\frac{\beta}{1+\beta}},$$

and therefore, for any $\hat{\epsilon} > 0$, provided $T_{\hat{\epsilon}} > 0$ is sufficiently small,

$$\|t^{1-\mu}\eta^{\frac{\beta}{1+\beta}} A(\eta I + A)^{-1} w_t\|_{E_0} \le \hat{\epsilon} \int_{\Gamma_{1,\psi}} |e^s s^{-\mu}||ds| = c\hat{\epsilon}, \tag{3.23}$$

for $0 < t \le T_{\hat{\epsilon}}$. Thus (3.21) is proved.

Write $u = v + w$, where u solves (3.9) and w is as in (3.22). Then v solves

$$D_t^\beta v_t + Av = 0, \quad t \in J, \quad v(0) = y, \quad v_t(0) = 0. \tag{3.24}$$

By (3.10) and by (3.21) we have (observe that (3.21), (3.22) imply $w \in \tilde{E}_1(J)$),

$$v = v_1 + v_2, \quad v_1 \in \tilde{E}_1(J), \quad v_2 \in \tilde{E}_{1,\epsilon}(J). \tag{3.25}$$

We assert that it follows from this that $y \in E_{\hat{\mu}}$.

To show this claim, first observe that by (3.25),

$$t^{1-\mu}\|Av(t)\|_{E_0} \to 0, \quad t \to 0. \tag{3.26}$$

So it suffices to show that (3.24), (3.26) imply $y \in E_{\hat{\mu}}$. Fix $\hat{\epsilon} > 0$. Then, for $T_{\hat{\epsilon}} > 0$ sufficiently small, and by (3.12), which holds with u replaced by v,

$$\begin{aligned} \|A\tilde{v}\|_{E_0} &= \|\int_0^\infty e^{-\lambda t} Av(t)\,dt\|_{E_0} \le \|\int_0^{T_{\hat{\epsilon}}} e^{-\lambda t} Av(t)\,dt\|_{E_0} \\ &+ \|\int_{T_{\hat{\epsilon}}}^\infty e^{-\lambda t} Av(t)\,dt\|_{E_0} \le \hat{\epsilon}\int_0^{T_{\hat{\epsilon}}} e^{-\lambda t} t^{\mu-1}\,dt + c\int_{T_{\hat{\epsilon}}}^\infty e^{-\lambda t}\,dt \le 2\hat{\epsilon}\Gamma(\mu)\lambda^{-\mu}, \end{aligned} \tag{3.27}$$

if $\lambda > 0$ is taken sufficiently large. Take transforms in (3.24) to get, after some simple manipulations, for $\lambda > 0$,

$$y = \lambda^{-\beta}(\lambda^{1+\beta} I + A)\tilde{v}.$$

Hence

$$\begin{aligned} &\lambda^{\mu+\beta} A(\lambda^{1+\beta} I + A)^{-1} y \\ &= \lambda^{\mu+\beta} A(\lambda^{1+\beta} I + A)^{-1} \lambda^{-\beta}(\lambda^{1+\beta} I + A)\tilde{v} = \lambda^\mu A\tilde{v}. \end{aligned} \tag{3.28}$$

By (3.27), (3.28) one has $y \in E_{\hat{\mu}}$. Recall (3.20) to see that Lemma 3.2 is proved. □

A combination of Lemmas 3.1 and 3.2 yields that we have shown that if $\varphi \in \tilde{E}_1(J)$, then $y \overset{\text{def}}{=} \varphi(0) \in E_{\hat{\mu}}$, $z \overset{\text{def}}{=} \varphi_t(0) \in E_{\tilde{\mu}}$.

To complete the proof of Theorem 2.4 we need to show that $(E_{\hat{\mu}} \times E_{\tilde{\mu}}) \subset \gamma(\tilde{E}_1(J))$.

For this it suffices to prove that if $y \in E_{\hat{\mu}}$, $z \in E_{\tilde{\mu}}$, then the solution u of

$$D_t^\beta(u_t - z) + Au = 0; \quad t > 0, \quad u(0) = y, \quad u_t(0) = z,$$

satisfies $u \in \tilde{E}_1(J)$.

By the arguments above, where (3.20) was assumed and (3.21) was proved, we may take $z = 0$. Thus it is enough to prove the following Lemma.

Lemma 3.3. *Let* $y \in E_{\frac{\mu+\beta}{1+\beta}}$ *and let* v *solve*

$$D_t^\beta v_t + Av = 0, \quad t > 0, \quad v(0) = y, \quad v_t(0) = 0.$$

Then

$$\lim_{t\to 0} t^{1-\mu}\|Av(t)\|_{E_0} = 0, \quad \lim_{t\to 0} t^{1-\mu}\|v_t\|_{E_{\frac{\beta}{1+\beta}}} = 0. \tag{3.29}$$

Proof. The first part of (3.29) follows as is the proof of [3, Theorem 7].

By the assumption on y, one has $v_t \in E_{\frac{\beta}{1+\beta}}$, for each $t > 0$. Then write

$$\begin{aligned} &t^{1-\mu}\eta^{\frac{\beta}{1+\beta}}A(\eta I+A)^{-1}v_t(t) \\ &= -(2\pi i)^{-1}t^{1-\mu}\int_{\Gamma_{1,\psi}} e^{\lambda t}\eta^{\frac{\beta}{1+\beta}}A(\eta I+A)^{-1}A(\lambda^{1+\beta}I+A)^{-1}y\,d\lambda \qquad (3.30)\\ &= -(2\pi i)^{-1}t^{-\mu}\int_{\Gamma_{1,\psi}} e^{s}\eta^{\frac{\beta}{1+\beta}}A^{\frac{1}{1+\beta}}(\eta I+A)^{-1}(\frac{s}{t})^{-\mu}(\frac{s}{t})^{\mu}A((\frac{s}{t})^{1+\beta}I+A)^{-1}A^{\frac{\beta}{1+\beta}}y\,ds. \end{aligned}$$

By the assumption on y, $A^{\frac{\beta}{1+\beta}}y \in E_{\frac{\mu}{1+\beta}}$. Hence

$$\lim_{t\to 0}\|(st^{-1})^{\mu}A((st^{-1})^{1+\beta}I + A)^{-1}A^{\frac{\beta}{1+\beta}}y\|_{E_0} = 0, \tag{3.31}$$

uniformly on $\Gamma_{1,\psi}$. Moreover,

$$\sup_{\eta>0}\|\eta^{\frac{\beta}{1+\beta}}A^{\frac{1}{1+\beta}}(\eta I + A)^{-1}\|_{L(E_0,E_0)} < \infty. \tag{3.32}$$

Use (3.31), (3.32) in (3.30) to obtain the second part of (3.29). Lemma 3.3 is proved; therefore also Theorem 2.2. □

4. Proof of Theorem 2.4

Split u in three parts; $u = v + w + \vartheta$, where v, w, ϑ satisfy, respectively,

$$\begin{aligned} &D_t^\beta v_t + Av = 0, \quad v(0) = y, \quad v_t(0) = 0,\\ &D_t^\beta(w_t - z) + Aw = 0, \quad w(0) = 0, \quad w_t(0) = z,\\ &D_t^\beta \vartheta_t + A\vartheta = f, \quad \vartheta(0) = 0, \quad \vartheta_t(0) = 0. \end{aligned}$$

First, handle the continuity of u.

By [3, Theorems 7 and 8, p. 428-430] we have $v \in C([0,T]; E_{\hat\mu})$ and $\|v(t)\|_{E_{\hat\mu}} \le c\|y\|_{E_{\hat\mu}}$.

Consider $w(t)$. One has, after employing the usual tricks,

$$\begin{aligned} \eta^{\tilde\mu}A(\eta I + A)^{-1}A^{\frac{1}{1+\beta}}w(t) &= (2\pi i)^{-1}\int_{\Gamma_{1,\psi}} e^{\lambda t}\lambda^{-1+\beta}F(\lambda)\eta^{\tilde\mu}A(\eta I + A)^{-1}z\,d\lambda\\ &=(2\pi i)^{-1}\int_{\Gamma_{1,\psi}} e^{s}(\frac{s}{t})^{-1+\beta}F(st^{-1})\eta^{\tilde\mu}A(\eta I + A)^{-1}zt^{-1}\,ds, \end{aligned}$$

where $F(\lambda) \stackrel{\text{def}}{=} A^{\frac{1}{1+\beta}}(\lambda^{1+\beta}I+A)^{-1}$. By the assumption on z, $\|\eta^{\tilde{\mu}}A(\eta I+A)^{-1}z\|_{E_0}$ can be taken arbitrarily small if η sufficiently large. Moreover, $\|F(st^{-1})\|_{L(E_0,E_0)} \leq c(st^{-1})^{-\beta}$. So, for any $\epsilon > 0$, if η sufficiently large, for some $c > 0$,

$$\|\eta^{\tilde{\mu}}A(\eta I+A)^{-1}A^{\frac{1}{1+\beta}}w(t)\|_{E_0} \leq c\epsilon, \tag{4.1}$$

uniformly for $t \in [0,T]$. Thus $A^{\frac{1}{1+\beta}}w(t) \in E_{\tilde{\mu}}$, $t > 0$, with uniformly bounded norm; hence $\sup_{t\in[0,T]}\|w(t)\|_{E_{\hat{\mu}}} < \infty$. The above estimates also give $\|w(t)\|_{E_{\hat{\mu}}} \leq c\|z\|_{E_{\tilde{\mu}}}$.

To prove continuity of $w(t)$ in $E_{\hat{\mu}}$, argue as follows.

Suppose we can show that $A^{\frac{1}{1+\beta}}w(t) \in C([0,T];E_0)$. We claim that then $w \in C([0,T];E_{\hat{\mu}})$, or, equivalently, $A^{\frac{1}{1+\beta}}w \in C([0,T];E_{\tilde{\mu}})$. Let $s,t \in [0,T]$. Then

$$\begin{aligned}\big\|A^{\frac{1}{1+\beta}}(w(t)-w(s))\big\|_{E_{\tilde{\mu}}} &= \sup_{\eta>0}\big\|\eta^{\tilde{\mu}}A(\eta I+A)^{-1}A^{\frac{1}{1+\beta}}(w(t)-w(s))\big\|_{E_0} \\ \leq 2 &\sup_{\eta>\eta_0;\tau\in[0,T]}\big\|\eta^{\tilde{\mu}}A(\eta I+A)^{-1}A^{\frac{1}{1+\beta}}w(\tau)\big\|_{E_0} \\ &+ \sup_{0<\eta\leq\eta_0}\|\eta^{\tilde{\mu}}A(\eta I+A)^{-1}\|_{L(E_0,E_0)}\|A^{\frac{1}{1+\beta}}(w(t)-w(s))\|_{E_0}.\end{aligned}$$

Take any $\epsilon > 0$. By (4.1) we can take η_0 so large that the first term on the right side above is $\leq \frac{\epsilon}{2}$. Fixing such a η_0 we then choose $\delta > 0$ so that if $|t-s| \leq \delta$, then the second term is $\leq \frac{\epsilon}{2}$. Thus $w \in C([0,T];E_{\hat{\mu}})$.

It remains to show that $A^{\frac{1}{1+\beta}}w(t) \in C([0,T];E_0)$. In particular, we need to show that $A^{\frac{1}{1+\beta}}w(t) \to 0$ in E_0 as $t \to 0$.

We have, by (2.16) and after the usual tricks,

$$A^{\frac{1}{1+\beta}}w(t) = (2\pi i)^{-1}\int_{\Gamma_{1,\psi}} e^s t^{-\beta}s^{-1+\beta}A^{\frac{1}{1+\beta}}\big((st^{-1})^{1+\beta}+A\big)^{-1}z\,ds.$$

Choose $\delta > 0$ so that $\rho \stackrel{\text{def}}{=} \mu+\beta-1-\delta > 0$. By assumption, $z \in E_{\tilde{\mu}}$. Then $A^{\frac{\delta}{1+\beta}}z \in E_{\frac{\rho}{1+\beta}}$, and after straightforward estimates (analogous to the estimates preceding (4.1)),

$$\|A^{\frac{1}{1+\beta}}w(t)\|_{E_{\frac{\rho}{1+\beta}}} \leq c\|A^{\frac{\delta}{1+\beta}}z\|_{E_{\frac{\rho}{1+\beta}}}t^{\delta}\int_{\Gamma_{1,\psi}} e^{-\gamma|s|}|s|^{-1-\delta}|ds|,$$

for some $\gamma, c > 0$. Thus $w \in C([0,T];E_{\hat{\mu}})$.

Consider then ϑ. By the assumption on maximal regularity in $\tilde{E}_0(J)$, $\vartheta \in BUC_{1-\mu}(J;E_1) \cap BUC^{1+\beta}_{1-\mu}(J;E_0)$. Now use interpolation arguments to conclude from this (see [3, p. 432]) that $\vartheta \in C([0,T];E_{\hat{\mu}})$. In addition, $\|\vartheta(t)\|_{E_{\hat{\mu}}} \leq c\|f\|_{\tilde{E}_0(J)}$, $t \in J$. We conclude that $u \in C([0,T];E_{\hat{\mu}})$ and satisfies the appropriate estimate in terms of the given data.

Consider u_t. Split u_t in the same manner as u and first take

$$v_t = -(2\pi i)^{-1}\int_{\Gamma_{1,\psi}} e^{\lambda t}A(\lambda^{1+\beta}I+A)^{-1}y\,d\lambda.$$

By analyticity and after the by now familiar change of variables,

$$\eta^{\tilde{\mu}}A(\eta I+A)^{-1}v_t=-(2\pi i)^{-1}\int_{\Gamma_{1,\psi}}s^{-1}e^sA^{\frac{\beta}{1+\beta}}(\frac{s}{t})((\frac{s}{t})^{1+\beta}I+A)^{-1}G(\eta)\,ds,\quad(4.2)$$

where $G(\eta)=\eta^{\tilde{\mu}}A(\eta I+A)^{-1}A^{\hat{\mu}-\tilde{\mu}}y$. We have $y\in E_{\hat{\mu}}$, thus $A^{\hat{\mu}-\tilde{\mu}}y\in E_{\tilde{\mu}}$. So $\|G(\eta)\|_{E_0}\to 0,\quad \eta\to\infty$. Use this and the uniform boundedness in s,t of $A^{\frac{\beta}{1+\beta}}(\frac{s}{t})((\frac{s}{t})^{1+\beta}I+A)^{-1}$ to get that the right side of (4.2) vanishes as $\eta\to\infty$, uniformly on $[0,T]$. Hence $v_t\in B([0,T];E_{\tilde{\mu}})$ and $\|v_t(t)\|_{E_{\tilde{\mu}}}\le c\|y\|_{E_{\hat{\mu}}}$.

Suppose we can show that $v_t\in C([0,T];E_0)$. Then argue as in the proof of $w\in C([0,T];E_{\hat{\mu}})$ and use the fact that the left side of (4.2) can be made arbitrarily small by choosing η sufficiently large, uniformly in t. This gives $v_t\in C([0,T];E_{\tilde{\mu}})$. We omit the details.

To show that actually $v_t\in C([0,T];E_0)$ and, in particular, that $\|v_t(t)\|_{E_0}\to 0$ as $t\to 0$, one argues in the same fashion as above where $A^{\frac{1}{1+\beta}}w\in C([0,T];E_0)$ was obtained. The details are left to the reader.

As to w_t, we write

$$w_t-z=-(2\pi i)^{-1}\int_{\Gamma_{1,\psi}}e^{\lambda t}\lambda^{-1}A(\lambda^{1+\beta}I+A)^{-1}z\,d\lambda.$$

Using the assumption on z, and the fact that $\|A(\lambda I+A)^{-1}\|_{L(E_0,E_0)}$ is uniformly bounded, one easily has from this

$$\|\eta^{\tilde{\mu}}A(\eta I+A)^{-1}[w_t-z]\|_{E_0}\to 0,\tag{4.3}$$

as $\eta\to\infty$; uniformly on $t\in[0,T]$. Also note that $\|w_t-z\|_{E_{\tilde{\mu}}}\le c\|z\|_{E_{\tilde{\mu}}}$; hence $\|w_t\|_{E_{\tilde{\mu}}}\le c\|z\|_{E_{\tilde{\mu}}}$. The function w_t is continuous on $[0,T]$ in E_0; see, e.g., the comment after (2.16). Use this fact, together with (4.3), to get $w_t\in C([0,T];E_{\tilde{\mu}})$. (Argue as in the proof of $w\in C([0,T];E_{\hat{\mu}})$.)

There remains to show that $\vartheta_t\in C([0,T];E_{\tilde{\mu}})$, in particular, to show that $\|\vartheta_t(t)\|_{E_{\tilde{\mu}}}\to 0,\quad t\to 0$.

We first claim

$$\sup_{t\in[0,T]}\|\vartheta_t(t)\|_{E_{\tilde{\mu}}}\le c\|f\|_{\tilde{E}_0(J)}.\tag{4.4}$$

To obtain this, use (3.6) and estimates analogous to those in the proof of Lemma 3.1. We leave the details to the reader.

Now observe that the same estimate (4.4) does hold for $J_\tau=[0,\tau]$, with $\tau<T$, and recall that

$$\lim_{\tau\to 0}\|f\|_{BUC_{1-\mu}([0,\tau];X)}=0.$$

Thus $\|\vartheta_t(t)\|_{E_{\tilde{\mu}}}\to 0,\quad t\to 0$.

To obtain (4.4), one may also argue as follows.

Recall that the equation governing ϑ is assumed to possess maximal regularity in $\tilde{E}_0(J)$. Hence $D_t^\beta\vartheta_t\in\tilde{E}_0(J)$, or, more specifically,

$$D_t^\beta\vartheta_t=t^{\mu-1}h(t),\tag{4.5}$$

where $h \in BUC_{0\to 0}(J;E_0)$, with $\sup_{t\in J}\|h(t)\|_{E_0} \leq c\|f\|_{\tilde{E}_0(J)}$. Convolve (4.5) with $t^{-1+\beta}$ and estimate, to get

$$\|\vartheta_t(t)\|_{E_0} \leq ct^{\beta+\mu-1}\|f\|_{\tilde{E}_0}. \tag{4.6}$$

Recall that we also have (Lemma 3.1 in the proof of Theorem 2.4),

$$\vartheta_t \in BUC_{1-\mu-\epsilon}(J;E_{\frac{\beta-\epsilon}{1+\beta}}), \quad \text{with} \quad \|\vartheta_t(t)\|_{E_{\frac{\beta-\epsilon}{1+\beta}}} \leq ct^{\epsilon+\mu-1}\|f\|_{\tilde{E}_0(J)}. \tag{4.7}$$

Interpolate between (4.6) and (4.7), recalling that

$$K(\tau,\vartheta_t(t),E_0,E_{\frac{\beta-\epsilon}{1+\beta}}) \stackrel{\text{def}}{=} \inf_{\vartheta_t=a+b}(\|a\|_{E_0}+\tau\|b\|_{E_{\frac{\beta-\epsilon}{1+\beta}}}).$$

Choose $a = \tau[\tau+t^{\beta-\epsilon}]^{-1}\vartheta_t(t)$; $b = t^{\beta-\epsilon}[\tau+t^{\beta-\epsilon}]^{-1}\vartheta_t(t)$. Then

$$K(\tau,\vartheta_t(t),E_0,E_{\frac{\beta-\epsilon}{1+\beta}}) \leq c\|f\|_{\tilde{E}_0(J)}\frac{\tau t^{\beta+\mu-1}}{\tau+t^{\beta-\epsilon}},$$

and

$$\|\vartheta_t(t)\|_{E_{\tilde{\mu}}} = \sup_{0\leq\tau\leq 1}\tau^{-\tilde{\mu}_\epsilon}K(\tau,\vartheta_t(t),E_0,E_{\frac{\beta-\epsilon}{1+\beta}}) \leq ct^{\beta+\mu-1}\|f\|_{\tilde{E}_0(J)}\sup_{0\leq\tau\leq 1}\frac{\tau^{1-\tilde{\mu}_\epsilon}}{\tau+t^{\beta-\epsilon}},$$

where $\tilde{\mu}_\epsilon = [\beta+\mu-1][\beta-\epsilon]^{-1}$. Now let $\tau = \sigma t^{\beta-\epsilon}$, then

$$\|\vartheta_t(t)\|_{E_{\tilde{\mu}}} \leq c\sup_{\sigma\in(0,\infty)}[\sigma+1]^{-1}[\sigma^{\frac{1-\epsilon-\mu}{\beta-\epsilon}}] \leq c.$$

Theorem 2.5 is proved.

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Semilinear Elliptic Problems Associated with the Complex Ginzburg-Landau Equation

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Dedicated to Professor Philippe Clément on the occasion of his official retirement

Abstract. The complex Ginzburg-Landau equation is characterized by complex coefficients with positive real parts in front of both the Laplacian and the nonlinear term. Let $\kappa+i\beta$ be the coefficient of the nonlinear term $|u|^{q-2}u$ with $q \ge 2$. If $\kappa^{-1}|\beta| \le 2\sqrt{q-1}/(q-2)$, then the family of solution operators forms a semigroup of (quasi-)contractions on $L^2(\Omega)$, $\Omega \subset \mathbb{R}^N$. Under an additional condition $2 \le q \le 2 + N/4$ the family forms a semigroup of locally Lipschitz operators even if $\kappa^{-1}|\beta| > 2\sqrt{q-1}/(q-2)$. As a beginning to understand such semigroups the resolvent problem for the equation is developed.

1. Introduction and result

This paper is mainly concerned with semilinear elliptic problems associated with the complex Ginzburg-Landau equation. First we try to explain the relationship between the two problems, elliptic and parabolic (stationary and non-stationary). Let us begin with initial-Dirichlet boundary value problems for the complex Ginzburg-Landau equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0 \ \text{ on } \ \Omega \times \mathbb{R}_+, \\ u = 0 \ \text{ on } \ \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{CGL}$$

where Ω is a bounded or unbounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}$) with C^2-boundary $\partial\Omega$. The equation is recognized with those complex coefficients in front of the Laplacian and nonlinear term so that the unknown function u is necessarily complex-valued. We assume as usual that $\lambda, \kappa \in \mathbb{R}_+ := (0,\infty)$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $q \ge 2$ are constants. Now we are in a position to write down elliptic (stationary) problems associated with (CGL):

$$\begin{cases} \zeta u - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-2}u - \gamma u = f \text{ on } \Omega, \\ u(x) = 0, \ \ x \in \partial\Omega, \end{cases} \tag{\textbf{StCGL}}$$

where ζ is a complex parameter with $\operatorname{Re}\zeta > \gamma$. The physical meaning of (**StCGL**) is not so clear that there seems to be no preceding investigation, while there are a lot of important articles on the well-posedness of (CGL) and related equations (by R. Temam, Y. Yang, C.D. Levermore-M. Oliver, J. Ginibre-G.Velo, S.-Z. Huang-P. Takač and others) which are collected in the References of the latest paper [11]. Moreover, a detailed comparison of the known results contained in the above-mentioned articles with those through a new abstract theory will be given in a forthcoming paper [12].

To study (CGL) and (**StCGL**) we employ an abstract formulation in a Hilbert space. In fact, (CGL) are regarded as abstract Cauchy problems for a semilinear evolution equation:

$$\begin{cases} \dfrac{du}{dt} + (\lambda + i\alpha)Su + (\kappa + i\beta)\partial\psi(u) - \gamma u = 0, \ \ t > 0, \\ u(0) = u_0. \end{cases} \tag{ACP}$$

Here S is the nonnegative selfadjoint operator in $X := L^2(\Omega)$ defined as $-\Delta$ with domain $D(S) := H^2(\Omega) \cap H_0^1(\Omega)$ (see Brézis [2, Théorème IX.25]), while $\partial\psi$ is the subdifferential of the lower semi-continuous convex function ψ on X defined as

$$\psi(u) := \begin{cases} (1/q)\|u\|_{L^q}^q & \text{if } \ u \in D(\psi) := L^2(\Omega) \cap L^q(\Omega), \\ \infty & \text{otherwise.} \end{cases} \tag{1.1}$$

As is well known, $\partial\psi$ is maximal monotone (m-accretive) in X, while so is $(\lambda+i\alpha)S$ for $\lambda \geq 0$. The monotonicity of $(\kappa + i\beta)\partial\psi$ depends on the ratio $|\beta|/\kappa$. It turns out that if

$$0 \leq \kappa^{-1}|\beta| \leq c_q^{-1} := (2\sqrt{q-1})/(q-2), \tag{1.2}$$

then $(\lambda + i\alpha)S + (\kappa + i\beta)\partial\psi$ is maximal monotone in X as the sum of two such operators. This is a *complex* generalization of the well-known theory in *real* Hilbert spaces (see, e.g., Clément-Egberts [3] and Egberts [4]). In fact, monotonicity methods together with the subdifferential viewpoint yield strong L^2-wellposedness of (CGL) including smoothing effect (see [9] and [10]). Namely, let $\{U(t); t \geq 0\}$ be the semigroup of quasi-contractions on X generated by

$$G := \gamma - (\lambda + i\alpha)S - (\kappa + i\beta)\partial\psi. \tag{1.3}$$

Then $\{U(t)\}$ has the following properties:

$$U(t)u_0 = \operatorname*{s-lim}_{n\to\infty}(1 - (t/n)G)^{-n}u_0 \ \ \forall\, t \geq 0, \ \ \forall\, u_0 \in X; \tag{1.4}$$

$$\|U(t)u_0 - U(t)v_0\| \leq e^{\gamma t}\|u_0 - v_0\| \ \ \forall\, t \geq 0, \ \ \forall\, u_0, v_0 \in X; \tag{1.5}$$

$$U(t)X \subset D(S) \cap D(\partial\psi) \ \ \forall\, t > 0.$$

Consequently, $u(t) := U(t)u_0$ is a unique strong solution to (ACP) with $u_0 \in X$. This means that if (1.2) is satisfied, then the solvability of (CGL) is reduced to the study of abstract stationary problems in X:

$$\zeta u + (\lambda + i\alpha)Su + (\kappa + i\beta)\partial\psi(u) - \gamma u = f, \tag{1.6}$$

where $\zeta = \xi + i\eta$ with $\xi > \gamma$. Roughly speaking, (1.4) and (1.5) are consequences of the estimate for the resolvent of G:

$$\|(\zeta - G)^{-1}f - (\zeta - G)^{-1}g\| \le (\xi - \gamma)^{-1}\|f - g\| \ \ \forall\, f, g \in X.$$

In this way our interest is naturally turned to the case in which (1.2) is replaced with

$$c_q^{-1} < |\beta|/\kappa < +\infty. \tag{1.7}$$

Recently, we have shown that the solution operators of (CGL) form a semigroup $\{U(t)\}$ of non-contraction operators on X though we have to impose a strong restriction (compared with the monotone case) on the power of nonlinearity:

$$2 \le q \le 2 + 4/N \tag{1.8}$$

(see [11] in which Ω is assumed to be bounded, and [12] for more general setting). In this case, the operator $-G$ defined by (1.3) is written as the sum $A + B$ of two operators:

$$A = (\lambda + i\alpha)S + (\kappa + i\delta c_q^{-1}\kappa)\partial\psi, \ \ B = i(\beta - \delta c_q^{-1}\kappa)\partial\psi - \gamma, \tag{1.9}$$

where $\delta := \operatorname{sgn}\beta$ so that A is maximal monotone and B is locally Lipschitz continuous in X, while $\{U(t)\}$ satisfies the estimate

$$\|U(t)u_0\| \le e^{\gamma t}\|u_0\| \ \ \forall\, t \ge 0, \ \ \forall\, u_0 \in X. \tag{1.10}$$

However, the factor $e^{\gamma t}$ in (1.5) becomes more complicated:

$$\|U(t)u_0 - U(t)v_0\| \le \exp\bigl\{K_1 t + K_2 e^{2\gamma_+ t}(\|u_0\| \vee \|v_0\|)^2\bigr\}\|u_0 - v_0\|, \tag{1.11}$$

where $K_1 := \gamma + (1-\theta)(|\beta| - c_q^{-1}\kappa)C_2$, $K_2 := (\theta/2q\kappa)(|\beta| - c_q^{-1}\kappa)C_2$ and

$$\theta = f_N(q) := \frac{2(q-2)}{2q - N(q-2)} \ \ \Bigl(2 \le q \le 2 + \frac{4}{N}\Bigr); \tag{1.12}$$

note that $0 = f_N(2) \le f_N(q) \le f_N(2 + 4/N) = 1$ (for C_2 see condition (**A5**) in Section 2 and Lemma 3.3). Nevertheless we notice that $K_1 \to \gamma$ and $K_2 \to 0$ as $|\beta| \to c_q^{-1}\kappa$, that is, (1.11) coincides with (1.5) in the limit. In this connection it might be meaningful to mention a special case. Suppose that $\gamma \le 0$, and introduce the ball $B_L := \{v \in X; \|v\| \le L\}$. Then (1.11) can be written as

$$\|U(t)u_0 - U(t)v_0\| \le Me^{K_1 t}\|u_0 - v_0\| \ \ \forall\, t \ge 0, \ \ \forall\, u_0, v_0 \in B_L, \tag{1.13}$$

where $\log M := K_2 L^2$. It follows from (1.13) and (1.10) with $\gamma \le 0$ that $\{U(t)\}$ is a Lipschitz semigroup on the closed subset B_L in the sense of Kobayashi-Tanaka [8]. The proof of these results has been carried out in the spirit of abstract semilinear evolution equations without having recourse to abstract stationary problem (1.6). In fact, no generation theorem is known for such kind of non-contraction semigroups satisfying (1.11).

The purpose of this paper is to show that the stationary problem (**StCGL**) under condition (1.7) is uniquely solvable in $L^2(\Omega)$ if among others the power q satisfies the "subcritical" condition

$$2 \le q < 2^* := \frac{2N}{(N-2)_+} = 1 + \frac{N+2}{(N-2)_+}. \tag{1.14}$$

Extend the function f_N defined as (1.12) from $[2, 2+4/N]$ to $[2, 2^*)$. Then we see that

$$1 = f_N\Big(2 + \frac{4}{N}\Big) \le f_N(q) = \frac{2(q-2)}{2N - q(N-2)} < \lim_{q \to 2^*} f_N(q) = \infty \ \text{ if } \ N \ge 2,$$

that is, θ is finite if $2 \le q < 2^*$; note that $f_1(q) \to 2$ $(q \to \infty)$. To solve (**StCGL**) we introduce a closed convex subset $C(M)$ in $L^2(\Omega)$ defined as

$$C(M) := \{w \in L^2(\Omega); \|w\|_{L^2} \le M, \ \|w\|_{L^q}^q \le M^2/\kappa\} \tag{1.15}$$

and consider (**StCGL**) with right-hand side $f \in C(M)$. Unfortunately, the class $C(M)$ of admissible right-hand sides is a proper subset of the ball B_M (though it is expected that $C(M)$ coincides with B_M itself if the power q satisfies (1.8) instead of (1.14)). Roughly speaking, we could show that $C(M)$ is invariant under the "resolvent" of $-G$ defined as (1.3), with some estimates for the iterations of the "resolvent". The proof is based on a combination of monotonicity and contraction principle. Anyway this might be a beginning of the first step toward the conceivable generation theory for the semigroups associated with (ACP) satisfying (1.7).

Before stating our result, we define a strong solution to (**StCGL**) as follows.

Definition 1.1. *A function $u \in L^2(\Omega)$ is said to be a strong solution to* (**StCGL**) *if*

(a) $u \in H^2(\Omega) \cap H_0^1(\Omega) \subset L^{2(q-1)}(\Omega)$;
(b) *u satisfies the equation in problem* (**StCGL**) *formulated in $L^2(\Omega)$.*

Now we state the main theorem in this paper.

Theorem 1.2. *Let $N \in \mathbb{N}$, $\lambda, \kappa \in \mathbb{R}_+$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Let $C(M)$ be the set defined as* (1.15)*. For $q \ge 2$ set $c_q := (q-2)/(2\sqrt{q-1})$. Assume that* (1.7) *and* (1.14) *are satisfied. If $|\alpha| \le c_q^{-1}\lambda$, then for every $f \in C(M)$ and $\zeta \in \mathbb{C}$ with*

$$\operatorname{Re}\zeta \ge 1 + \gamma + (|\beta| - c_q^{-1}\kappa)(C_2 \vee C_3)\Big(\frac{M^2}{q\kappa}\Big)^{\theta} \tag{1.16}$$

there exists a unique strong solution $u = u(\zeta) \in C(M)$ to (**StCGL**) *such that*

(a) $\|u\|_{L^2} \le \dfrac{1}{\operatorname{Re}\zeta}\|f\|_{L^2} \le M \ \ \forall\, f \in C(M)$;
(b) $\|\nabla u\|_{L^2}^2 \le \dfrac{1}{\lambda \operatorname{Re}\zeta}\|f\|_{L^2}^2 \le \dfrac{M^2}{\lambda}, \ \ \|u\|_{L^q}^q \le \dfrac{1}{\kappa \operatorname{Re}\zeta}\|f\|_{L^2}^2 \le \dfrac{M^2}{\kappa} \ \ \forall\, f \in C(M)$;
(c) $\|Su\|_{L^2} \le \dfrac{2}{\lambda}\|f\|_{L^2} \le \dfrac{2M}{\lambda}, \ \ \|u\|_{L^{2(q-1)}} \le \dfrac{\|f\|_{L^2}}{\kappa} \le \dfrac{M}{\kappa} \ \ \forall\, f \in C(M)$.

Here C_2 and C_3 are constants depending on $\lambda, \kappa, \beta, \gamma, q$ and N. Furthermore, let $v = v(\zeta)$ be another strong solution to (**StCGL**) *with right-hand side $g \in C(M)$. Then*

$$\|u - v\|_{L^2} \leq \{\xi - \gamma - (|\beta| - c_q^{-1}\kappa)C_2(M^2/q\kappa)^{\theta}\}^{-1}\|f - g\|_{L^2}. \tag{1.17}$$

Remark 1.3. The critical case of $q = 2^* = 1 + (N+2)/(N-2)$ for $N \geq 3$ may be included if right-hand sides are taken from balls in $H_0^1(\Omega)$ instead of $L^q(\Omega)$ (see (3.4) with $a = 1$).

Now let G be the operator defined as (1.3) with domain $D(G) := H^2(\Omega) \cap H_0^1(\Omega)$. Then we can define the restriction G_M of G to $D(G_M) := D(S) \cap C(M)$.

Corollary 1.4. *Under the setting of* Theorem 1.2 *$\zeta - G_M$ is invertible for $\zeta \in \mathbb{C}$ satisfying* (1.16)*, with range $R(\zeta - G_M) \supset C(M)$ and*

$$\|(\zeta - G_M)^{-n}f - (\zeta - G_M)^{-n}g\|_{L^2} \leq \{\xi - \gamma - (|\beta| - c_q^{-1}\kappa)C_2(M^2/q\kappa)^{\theta}\}^{-n}\|f - g\|_{L^2}$$

for every f, $g \in C(M)$ and $n \in \mathbb{N}$.

2. Abstract stationary problem

In this section we shall develop an abstract theory of semilinear elliptic equations which will be successfully applied to our problem (**StCGL**) in the next section.

Let X be a complex Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let S be a nonnegative selfadjoint operator with domain $D(S)$ in X. Let $\psi : X \to (-\infty, \infty]$ be a lower semi-continuous convex function, with $D(\psi) := \{u \in X;\, \psi(u) < \infty\} \neq \emptyset$. Then the subdifferential of ψ at $u \in D(\psi)$ is in general defined as

$$\partial\psi(u) := \{f \in X;\, \psi(v) - \psi(u) \geq \mathrm{Re}(f, v - u) \ \ \forall\, v \in D(\psi)\}.$$

Here we assume for simplicity that $\psi \geq 0$ and $\partial\psi$ is single-valued. It is worth noticing that S is also represented by a subdifferential: $S = \partial\varphi$, where φ is a convex function on X defined as

$$\varphi(u) := \begin{cases} (1/2)\|S^{1/2}u\|^2 & \text{if } \ u \in D(\varphi) := D(S^{1/2}), \\ \infty & \text{otherwise.} \end{cases} \tag{2.1}$$

Under this notation we consider **Abstract Stationary Problems** of the form:

$$(\xi + i\eta)u + (\lambda + i\alpha)Su + (\kappa + i\beta)\partial\psi(u) = f, \tag{\textbf{AStP}}$$

where $\xi, \lambda, \kappa \in \mathbb{R}_+$ and $\eta, \alpha, \beta \in \mathbb{R}$ are constants (for simplicity we have assumed that $\gamma = 0$ in (1.6) in Section 1). To write down the assumption imposed on (**AStP**) we need the *Yosida approximation* $(\partial\psi)_\varepsilon$ of the subdifferential $\partial\psi$:

$$(\partial\psi)_\varepsilon := \frac{1}{\varepsilon}(1 - J_\varepsilon), \quad J_\varepsilon = J_\varepsilon(\partial\psi) := (1 + \varepsilon\partial\psi)^{-1} \ \ \forall\, \varepsilon > 0.$$

As is well known we have the formal "associative law" $(\partial\psi)_\varepsilon = \partial(\psi_\varepsilon)$, where ψ_ε is the *Moreau-Yosida regularization* of ψ defined as

$$\psi_\varepsilon(v) := \min_{w\in X}\Big\{\psi(w) + \frac{1}{2\varepsilon}\|w-v\|^2\Big\}, \ \ \forall\, v \in X, \ \ \forall\, \varepsilon > 0$$

(see Brézis [1, Proposition 2.11] or Showalter [13, Proposition IV.1.8]). Thus we can use the simplified notation $\partial\psi_\varepsilon := (\partial\psi)_\varepsilon = \partial(\psi_\varepsilon)$, with

$$\psi(J_\varepsilon v) \le \psi_\varepsilon(v) = \psi(J_\varepsilon v) + \frac{\varepsilon}{2}\|\partial\psi_\varepsilon(v)\|^2 \le \psi(v). \tag{2.2}$$

Now we introduce the following seven conditions on S and ψ:

(**A1**) $\exists\, q \in [2,\infty)$ such that $\psi(\zeta u) = |\zeta|^q\psi(u)$ for $u \in D(\psi)$ and $\zeta \in \mathbb{C}$ with $\mathrm{Re}\,\zeta > 0$.

(**A2**) $\partial\psi$ is sectorially-valued in the sense of Kato [6]: $\exists\, c_q > 0$ such that

$$|\mathrm{Im}(\partial\psi(u) - \partial\psi(v), u - v)| \le c_q\mathrm{Re}(\partial\psi(u) - \partial\psi(v), u - v) \ \ \forall\, u, v \in D(\partial\psi).$$

In other words, $\partial\psi$ is anglebounded (see Egberts [4]).

(**A3**) For the same constant c_q as in (**A2**),

$$|\mathrm{Im}(Su, \partial\psi_\varepsilon(u))| \le c_q\mathrm{Re}(Su, \partial\psi_\varepsilon(u)) \ \ \forall\, u \in D(S).$$

(**A4**) $D(S) \subset D(\partial\psi)$ and $\exists\, C_1 > 0$ such that $\|\partial\psi(u)\| \le C_1(\|u\| + \|Su\|)$ for $u \in D(S)$.

(**A5**) $\forall\, \rho > 0\ \exists\, C_2 = C_2(\rho) > 0$ such that for $u, v \in D(\varphi)$ and $\varepsilon > 0$,

$$|\mathrm{Im}(\partial\psi_\varepsilon(u) - \partial\psi_\varepsilon(v), u - v)| \le \rho\varphi(u - v) + C_2\bigg(\frac{\psi(J_\varepsilon u) + \psi(J_\varepsilon v)}{2}\bigg)^{\theta}\|u - v\|^2,$$

where $\theta \ge 0$ is a constant.

(**A6**) $\forall\, \rho > 0\ \exists\, C_3 = C_3(\rho) > 0$ such that for $u \in D(S)$ and $\varepsilon > 0$,

$$|\mathrm{Im}(Su, \partial\psi_\varepsilon(u))| \le \rho\|Su\|^2 + C_3\psi(u)^\theta\varphi(u),$$

where $\theta \ge 0$ is the same constant as in (**A5**).

(**A7**) $\exists\, C_4 > 0$ such that for $u, v \in D(\partial\psi)$ and $\nu, \mu > 0$,

$$|\mathrm{Im}(\partial\psi_\nu(u) - \partial\psi_\mu(u), v)| \le C_4|\nu - \mu|\big(\sigma\|\partial\psi(u)\|^2 + \tau\|\partial\psi(v)\|^2\big),$$

where $\sigma, \tau > 0$ are constants satisfying $\sigma + \tau = 1$.

Remark 2.1. We need also (**A5**) with $\psi(J_\varepsilon u)$ and $\psi(J_\varepsilon v)$ replaced with $\psi(u)$ and $\psi(v)$, respectively:

(**A5**)$'$ $\forall\, \rho > 0\ \exists\, C_2 = C_2(\rho) > 0$ such that for $u, v \in D(\varphi) \cap D(\psi)$ and $\varepsilon > 0$,

$$|\mathrm{Im}(\partial\psi_\varepsilon(u) - \partial\psi_\varepsilon(v), u - v)| \le \rho\varphi(u - v) + C_2\bigg(\frac{\psi(u) + \psi(v)}{2}\bigg)^{\theta}\|u - v\|^2.$$

As is easily seen from (2.2), we have (**A5**) $\Rightarrow$ (**A5**)$'$. Moreover, letting ε tend to zero in condition (**A5**)$'$, we have

$(\mathbf{A5})''$ $\forall\,\rho > 0\ \exists\, C_2 = C_2(\rho) > 0$ such that for $u, v \in D(\varphi) \cap D(\partial\psi)$,

$$|\mathrm{Im}(\partial\psi(u) - \partial\psi(v), u - v)| \leq \rho\varphi(u - v) + C_2\left(\frac{\psi(u) + \psi(v)}{2}\right)^{\theta} \|u - v\|^2.$$

Given $f \in X$, we try to find a (unique) strong solution to (**AStP**) the definition of which is given as follows.

Definition 2.2. *A function $u \in X$ is called a strong solution to* (**AStP**) *if*

(a) $u \in D(S) \cap D(\partial\psi)$;

(b) *u satisfies equation in Abstract Stationary Problem* (**AStP**) *in X.*

Note that under condition (**A4**) (a) in Definition 2.2 should be replaced with

(a)′ $u \in D(S) \subset D(\partial\psi)$.

Now we state the main result in this section. The following theorem is a combined version of the old and new results so that we can compare both of them.

Theorem 2.3. *Let $\lambda, \kappa \in \mathbb{R}_+$, $\alpha, \beta \in \mathbb{R}$ and $\zeta = \xi + i\eta$ with $\xi > 0 = \gamma$, $\eta \in \mathbb{R}$.*

(**I**) (Monotone nonlinearity) *Assume that $|\beta| \leq c_q^{-1}\kappa$ and* (**A1**)–(**A3**) *are satisfied. Then $(\lambda + i\alpha)S + (\kappa + i\beta)\partial\psi$ is maximal monotone in X and hence for every $f \in X$ there exists a unique strong solution $u = u(\zeta) \in D(S) \cap D(\partial\psi)$ to* (**AStP**) *such that*

(a) $\|u(\zeta)\| \leq \xi^{-1}\|f\|\ \forall\,\xi > 0$;

(b) $2\lambda\varphi(u(\zeta)) + q\kappa\psi(u(\zeta)) \leq \xi^{-1}\|f\|^2\ \forall\,\xi > 0$;

(c) $\|Su(\zeta)\| \leq \lambda^{-1}\|f\|$;

(d) $\varphi(u(\zeta)) \leq \xi^{-2}\varphi(f)\ \forall\, f \in D(\varphi),\ \forall\,\xi > 0$.

Furthermore, let v be another solution to (**AStP**) *with f replaced with $g \in X$. Then*

$$\|u(\zeta) - v(\zeta)\| \leq \xi^{-1}\|f - g\|\ \ \forall\,\xi > 0. \tag{2.3}$$

(**II**) (Non-monotone nonlinearity) *Assume that*

$$|\beta| > c_q^{-1}\kappa, \quad |\alpha| \leq c_q^{-1}\lambda$$

and conditions (**A1**)$-$(**A7**) *are satisfied. Then for every $f \in C(M)$ and $\zeta \in \mathbb{C}$ with*

$$\xi = \mathrm{Re}\,\zeta \geq 1 + (\tilde{C}_2 \vee \tilde{C}_3)\left(\frac{M^2}{q\kappa}\right)^{\theta} \tag{2.4}$$

there exists a unique strong solution $u = u(\zeta) \in D(S) \cap C(M)$ to (**AStP**) *such that*

(a) $\|u(\zeta)\| \leq \xi^{-1}\|f\| \leq M\ \forall\, f \in C(M)$;

(b) $2\lambda\varphi(u(\zeta)) + q\kappa\psi(u(\zeta)) \leq \xi^{-1}\|f\|^2 \leq M^2\ \forall\, f \in C(M)$;

(c) $\|Su(\zeta)\| \leq \dfrac{2}{\lambda}\|f\| \leq \dfrac{2M}{\lambda},\ \ \|\partial\psi(u(\zeta))\| \leq \dfrac{\|f\|}{\kappa} \leq \dfrac{M}{\kappa}\ \ \forall\, f \in C(M)$.

Here $C(M)$ is a closed convex subset in X defined as

$$C(M) := \{w \in X; \|w\| \leq M,\ \psi(w) \leq M^2/(q\kappa)\} \tag{2.5}$$

and $\tilde{C}_j := (|\beta| - c_q^{-1}\kappa)C_j$ $(j = 2, 3)$*; for the constants* C_2*,* C_3 *see condition* (**A5**)*,* (**A6**)*.*

Furthermore, let v *be a strong solution to* (**AStP**) *with right-hand side* $g \in C(M)$*. Then*

$$\|u(\zeta) - v(\zeta)\| \le \left\{\xi - \tilde{C}_2\left(\frac{M^2}{q\kappa}\right)^\theta\right\}^{-1}\|f - g\| \;\; \forall\, \xi \ge 1 + (\tilde{C}_2 \vee \tilde{C}_3)\left(\frac{M^2}{q\kappa}\right)^\theta. \quad (2.6)$$

Remark 2.4. In Part (**II**) $(\tilde{C}_2 \vee \tilde{C}_3)(M^2/q\kappa)^\theta \to 0$ as $|\beta| \to c_q^{-1}\kappa$, that is, (2.3) may be regarded as the limit of (2.6).

Now let G be the operator defined as (1.3) with domain $D(G) := D(S) \subset D(\partial\psi)$. Then we can define the restriction G_M of G to $D(G_M) := D(S) \cap C(M)$.

Corollary 2.5. *Under the setting of* Theorem 2.1 Part (II)*,* $\zeta - G_M$ *is invertible for* $\zeta \in \mathbb{C}$ *satisfying* (2.4)*, with range* $R(\zeta - G_M) \supset C(M)$ *and*

$$\|(\zeta - G_M)^{-n}f - (\zeta - G_M)^{-n}g\| \le \{\xi - \tilde{C}_2(M^2/q\kappa)^\theta\}^{-n}\|f - g\|$$

for every f*,* $g \in C(M)$ *and* $n \in \mathbb{N}$*.*

The conclusion of Part (**I**) is contained in [10, Theorem 5.2] (in which S is assumed to be a (nonlinear) subdifferential operator in X). Therefore it remains to prove Part (**II**). Let $\lambda, \kappa \in \mathbb{R}_+$, $\alpha, \beta \in \mathbb{R}$. In what follows we assume that $\kappa^{-1}|\beta| \in (c_q^{-1}, \infty)$. Given $\varepsilon > 0$, we consider the following problem approximate to (**AStP**):

$$(\xi + i\eta)u + (\lambda + i\alpha)Su_\varepsilon + (\kappa + i\delta c_q^{-1}\kappa)\partial\psi(u_\varepsilon) + i(\beta - \delta c_q^{-1}\kappa)\partial\psi_\varepsilon(u_\varepsilon) = f.$$

Here we choose δ in such a way that $|\beta - \delta c_q^{-1}\kappa| = |\beta| - c_q^{-1}\kappa$ (> 0), that is,

$$\delta := \operatorname{sgn}\beta \;\; [= +1\ (\beta > 0),\ -1\ (\beta < 0)]$$

so that $|\delta| = 1$. Then it follows from [10, Theorem 5.1] that $(\lambda + i\alpha)S + (\kappa + i\delta c_q^{-1}\kappa)\partial\psi$ is maximal monotone in X (as mentioned in Part (**I**); note that $|\mathrm{Im}(\kappa + i\delta c_q^{-1}\kappa)| = c_q^{-1}\mathrm{Re}(\kappa + i\delta c_q^{-1}\kappa)$). On the other hand, $\partial\psi_\varepsilon$ is Lipschitz continuous on X:

$$\|\partial\psi_\varepsilon(u) - \partial\psi_\varepsilon(v)\| \le \varepsilon^{-1}\|u - v\| \;\; \forall\, u, v \in X$$

(see [1, Proposition 2.6] or [13, Theorem IV.1.1]). Setting (see (1.9))

$$A := (\lambda + i\alpha)S + (\kappa + i\delta c_q^{-1}\kappa)\partial\psi,\;\; D(A) := D(S) \cap D(\partial\psi),$$

$$B_\varepsilon := i(\beta - \delta c_q^{-1}\kappa)\partial\psi_\varepsilon,\;\; D(B_\varepsilon) := X,$$

the approximate equation is simply written as

$$\zeta u_\varepsilon + Au_\varepsilon + B_\varepsilon u_\varepsilon = f. \quad (\mathbf{AStP})_\varepsilon$$

Here we note that for every $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\frac{|\beta| - c_q^{-1}\kappa}{\varepsilon} < n_0. \quad (2.7)$$

The left-hand side is nothing but the Lipschitz constant of B_ε.

Consequently, we have

Lemma 2.6. *Assume that conditions* (**A1**)–(**A3**) *are satisfied. Then for every* $f \in X$ *there exists a unique solution* $u_\varepsilon = u_\varepsilon(\zeta) \in D(A)$ *to* $(\mathbf{AStP})_\varepsilon$. *Here* $\operatorname{Re}\zeta \geq n_0(\varepsilon)$, *that is, the resolvent set of* $-(A+B_\varepsilon)$ *contains the right half plane depending on* ε.

In fact, using the monotonicity of A, we can define the mapping on X:

$$T_0 w := (\zeta + A)^{-1}(f - B_\varepsilon w) \;\; \forall \operatorname{Re}\zeta > 0.$$

It then follows from (2.3) and the Lipschitz continuity of B_ε that

$$\|T_0 w - T_0 z\| \leq \frac{1}{\operatorname{Re}\zeta}\|B_\varepsilon w - B_\varepsilon z\| \leq \frac{|\beta| - c_q^{-1}\kappa}{\varepsilon \operatorname{Re}\zeta}\|w - z\| \;\; \forall\, w, z \in X.$$

In view of (2.7) we see that T_0 becomes a strict contraction on X if $\operatorname{Re}\zeta \geq n_0(\varepsilon)$.

Next we want to prove that the region of the parameter ζ for which the solvability of $(\mathbf{AStP})_\varepsilon$ is independent of ε is determined by the size of the right-hand side f.

For $\varepsilon > 0$ fix a closed convex subset in X:

$$C_\varepsilon(L) := \{w \in X; \psi_\varepsilon(w) \leq L\} \neq \emptyset, \tag{2.8}$$

where $L > 0$ is a constant; note that $\psi_\varepsilon \in C^1(X, \mathbb{R})$ is convex.

The next proposition is concerned with the unique solvability of $(\mathbf{AStP})_\varepsilon$ with right-hand side from $C_\varepsilon(L)$ under an additional condition (**A5**).

Proposition 2.7. *Let* $f \in C_\varepsilon(L)$ *and* $\lambda^{-1}|\alpha| \leq c_q^{-1}$. *Assume that conditions* (**A1**) – (**A3**) *and* (**A5**) *are satisfied. Then for every* $\zeta \in \mathbb{C}$ *with*

$$\xi := \operatorname{Re}\zeta \geq 1 + \tilde{C}_2 L^\theta$$

there exists a unique solution $u_\varepsilon = u_\varepsilon(\zeta) \in D(A)$ *to* $(\mathbf{AStP})_\varepsilon$. *Here* $\tilde{C}_2 := (|\beta| - c_q^{-1}\kappa)C_2$ *and* C_2 *is the constant in condition* (**A5**).

To prove Proposition 2.7 it is useful to summarize several properties of $\partial\psi$ and $\partial\psi_\varepsilon$ guaranteed by conditions (**A1**) and (**A2**).

Lemma 2.8. *Under condition* (**A1**) *one has*

(a) $\operatorname{Re}(\partial\psi(u), u) = q\,\psi(u) \;\; \forall\, u \in D(\partial\psi)$;

(b) $\operatorname{Im}(\partial\psi(u), u) = 0 \;\; \forall\, u \in D(\partial\psi)$;

(c) $\operatorname{Re}(\partial\psi_\varepsilon(v), v) \geq q\,\psi(J_\varepsilon v) \;\; \forall\, v \in X$, *where* $J_\varepsilon := (1 + \varepsilon\,\partial\psi)^{-1}$;

(d) $\operatorname{Im}(\partial\psi_\varepsilon(v), v) = 0 \;\; \forall\, v \in X$.

Moreover, under condition (**A2**) *one has*

(e) $|\operatorname{Im}(\partial\psi(u), \partial\psi_\varepsilon(u))| \leq c_q \operatorname{Re}(\partial\psi(u) - \partial\psi_\varepsilon(u), \partial\psi_\varepsilon(u))$ *and hence*

$$|\operatorname{Im}(\partial\psi(u), \partial\psi_\varepsilon(u))| \leq c_q \operatorname{Re}(\partial\psi(u), \partial\psi_\varepsilon(u)) \;\; \forall\, u \in D(\partial\psi).$$

Proof. (a), (b) and (d) are proved in [10, Lemma 3.2].

(c) First we note that $\partial\psi_\varepsilon(v) = \partial\psi(J_\varepsilon v)$. In fact, put $v_\varepsilon := J_\varepsilon v = (1 + \varepsilon\partial\psi)^{-1}v$. Then we have $v_\varepsilon + \varepsilon\partial\psi(v_\varepsilon) = v = v_\varepsilon + \varepsilon\partial\psi_\varepsilon(v)$; the second equality is nothing but the definition of $\partial\psi_\varepsilon$. Thus we obtain

$$\begin{aligned}\operatorname{Re}(\partial\psi_\varepsilon(v), v) &= \operatorname{Re}(\partial\psi_\varepsilon(v), v - J_\varepsilon v + J_\varepsilon v)\\ &= \varepsilon\|\partial\psi_\varepsilon(v)\|^2 + \operatorname{Re}(\partial\psi(J_\varepsilon v), J_\varepsilon v)\\ &\geq q\,\psi(J_\varepsilon v).\end{aligned}$$

In the last step we have used (a) with $u := J_\varepsilon v$.

(e) In the same way as in (c) we have

$$\operatorname{Re}(\partial\psi(u) - \partial\psi_\varepsilon(u), \partial\psi_\varepsilon(u)) = \varepsilon^{-1}\operatorname{Re}(\partial\psi(u) - \partial\psi(J_\varepsilon u), u - J_\varepsilon u).$$

Applying condition (**A2**) to the right-hand side, we see that it is greater than or equal to

$$\begin{aligned}&(1/c_q)\varepsilon^{-1}|\operatorname{Im}(\partial\psi(u) - \partial\psi(J_\varepsilon u), u - J_\varepsilon u))|\\ &\quad= (1/c_q)|\operatorname{Im}(\partial\psi(u) - \partial\psi(J_\varepsilon u), (u - J_\varepsilon u)/\varepsilon) + \operatorname{Im}(\partial\psi(J_\varepsilon u), (u - J_\varepsilon u)/\varepsilon)|\\ &\quad= (1/c_q)|\operatorname{Im}(\partial\psi(u), \partial\psi_\varepsilon(u))|;\end{aligned}$$

note that $\operatorname{Im}(\partial\psi(J_\varepsilon u), (u - J_\varepsilon u)/\varepsilon) = \operatorname{Im}\|\partial\psi_\varepsilon(u)\|^2 = 0$. □

Now we begin the proof of Proposition 2.7 with $\zeta = \xi + i\eta$. For $\xi < n_0$ we try to solve a modified form of equation $(\mathbf{AStP})_\varepsilon$:

$$(n_0 + i\eta)u_\varepsilon + Au_\varepsilon + B_\varepsilon u_\varepsilon = f + (n_0 - \xi)u_\varepsilon.$$

Since $(n_0 + i\eta + A + B_\varepsilon)^{-1}$ is well-defined on X (see Lemma 2.6), we can introduce the mapping T on the closed convex subset $C_\varepsilon(L)$ defined by (2.8):

$$Tw := \Big(\frac{n_0 + i\eta}{n_0} + \frac{1}{n_0}(A + B_\varepsilon)\Big)^{-1}\Big[\frac{1}{n_0}f + \frac{n_0 - \xi}{n_0}w\Big] \quad \forall\, w \in C_\varepsilon(L).$$

Then the proof of Proposition 2.7 is divided into two parts which respectively show that $T\,C_\varepsilon(L) \subset C_\varepsilon(L)$ (Lemma 2.9) and that T is a strict contraction on $C_\varepsilon(L)$ (Lemma 2.11).

Lemma 2.9. *Let $f \in C_\varepsilon(L)$ and $\lambda^{-1}|\alpha| \leq c_q^{-1}$. Assume that conditions* (**A1**)–(**A3**) *are satisfied. If $\xi = \operatorname{Re}\zeta \geq 1$, then $Tw \in C_\varepsilon(L)$ for every $w \in C_\varepsilon(L)$.*

Proof. In view of (2.8) it suffices to show that $\psi_\varepsilon(Tw) \leq L$.

For $f, w \in C_\varepsilon(L)$ put $z := Tw \in D(A) = D(S) \cap D(\partial\psi)$. Then by definition we have

$$\Big(1 + i\frac{\eta}{n_0}\Big)z + \frac{\lambda + i\alpha}{n_0}Sz + \frac{\kappa + i\delta c_q^{-1}\kappa}{n_0}\partial\psi(z) + \frac{1}{n_0}B_\varepsilon z = \frac{1}{n_0}f + \frac{n_0 - \xi}{n_0}w, \quad (2.9)$$

where $1 \le \xi < n_0$. It follows from the convexity of ψ_ε that

$$\begin{aligned}\psi_\varepsilon\left(\frac{1}{n_0}f + \frac{n_0-\xi}{n_0}w\right) &\le \frac{\xi}{n_0}\psi_\varepsilon\left(\frac{f}{\xi}\right) + \frac{n_0-\xi}{n_0}\psi_\varepsilon(w)\\ &\le \frac{\xi}{n_0}\left(\frac{1}{\xi}\right)^2\psi_\varepsilon(f) + \frac{n_0-\xi}{n_0}\psi_\varepsilon(w);\end{aligned}$$

note that $\psi_\varepsilon(\zeta f) \le |\zeta|^2\psi_\varepsilon(f)$ for $\zeta \in \mathbb{C}$ with $|\zeta| \le 1$ (by definition of ψ_ε and condition (**A1**)). Since $\xi \ge 1$, we see that

$$\psi_\varepsilon\left(\frac{1}{n_0}f + \frac{n_0-\xi}{n_0}w\right) \le \frac{\xi}{n_0}L + \frac{n_0-\xi}{n_0}L = L. \tag{2.10}$$

Taking the real part of the inner product of (2.9) with $\partial\psi_\varepsilon(z)$, we have

$$\begin{aligned}\mathrm{Re}(z,\partial\psi_\varepsilon(z)) \;+\; &\mathrm{Re}\left\{\frac{\lambda+i\alpha}{n_0}(Sz,\partial\psi_\varepsilon(z))\right\}\\ &+\mathrm{Re}\left\{\frac{\kappa+i\delta c_q^{-1}\kappa}{n_0}(\partial\psi(z),\partial\psi_\varepsilon(z))\right\} \qquad (2.11)\\ = \mathrm{Re}&\left(\frac{1}{n_0}f + \frac{n_0-\xi}{n_0}w, \partial\psi_\varepsilon(z)\right) \;=: \mathrm{IP};\end{aligned}$$

note that $(B_\varepsilon z, \partial\psi_\varepsilon(z))$ is purely imaginary. In view of (2.10) IP is computed by the definition of subdifferential as follows:

$$\begin{aligned}\mathrm{IP} =\; &\mathrm{Re}\Big(\frac{1}{n_0}f + \frac{n_0-\xi}{n_0}w - z, \partial\psi_\varepsilon(z)\Big) + \mathrm{Re}\,(z,\partial\psi_\varepsilon(z))\\ &\le\; \psi_\varepsilon\left(\frac{1}{n_0}f + \frac{n_0-\xi}{n_0}w\right) - \psi_\varepsilon(z) + \mathrm{Re}\,(z,\partial\psi_\varepsilon(z))\\ \le\; &L - \psi_\varepsilon(z) + \mathrm{Re}\,(z,\partial\psi_\varepsilon(z)).\end{aligned}$$

On the other hand, we see from condition (**A3**) and Lemma 2.8 (e) that

$$\begin{aligned}\mathrm{Re}\left\{\frac{\lambda+i\alpha}{n_0}(Sz,\partial\psi_\varepsilon(z))\right\} &\ge \frac{c_q^{-1}\lambda - |\alpha|}{n_0}|\mathrm{Im}(Sz,\partial\psi_\varepsilon(z))|;\\ \mathrm{Re}\left\{\frac{\kappa+i\delta c_q^{-1}\kappa}{n_0}(\partial\psi(z),\partial\psi_\varepsilon(z))\right\} &\ge \frac{c_q^{-1}\kappa - |\delta c_q^{-1}\kappa|}{n_0}|\mathrm{Im}(\partial\psi(z),\partial\psi_\varepsilon(z))| = 0.\end{aligned}$$

In this way (2.11) leads us to the desired inequality

$$\psi_\varepsilon(z) + \frac{c_q^{-1}\lambda - |\alpha|}{n_0}|\mathrm{Im}(Sz,\partial\psi_\varepsilon(z))| \le L.$$

Thus we need the restriction $|\alpha| \le c_q^{-1}\lambda$ to conclude that $Tw = z \in C_\varepsilon(L)$. □

Remark 2.10. Let $f \in B_M = \{f \in X; \|f\| \le M\}$ in Lemma 2.9. Then B_M is invariant under the mapping T.

Lemma 2.11. *Let $f \in C_\varepsilon(L)$ and $\lambda^{-1}|\alpha| \le c_q^{-1}$. Assume that conditions* (**A1**)−(**A3**) *and* (**A5**) *are satisfied. If $\xi = \operatorname{Re}\zeta \ge 1 + \tilde{C}_2 L^\theta$, then T is a strict contraction on $C_\varepsilon(L)$.*

Proof. It suffices to show that if $1 \le \xi < n_0$, then

$$\|Tw - Tz\| \le (n_0 - \xi)(n_0 - \tilde{C}_2 L^\theta)^{-1}\|w - z\| \ \ \forall\, w, z \in C_\varepsilon(L). \tag{2.12}$$

In fact, the coefficient on the right-hand side is less than 1 if $\xi \ge 1 + \tilde{C}_2 L^\theta$.

We begin with the estimate on the resolvent $(1+i(\eta/n_0)+(1/n_0)(A+B_\varepsilon))^{-1}$. Put

$$g := \frac{1}{n_0}f + \frac{n_0 - \xi}{n_0}w, \quad h := \frac{1}{n_0}f + \frac{n_0 - \xi}{n_0}z \quad (w, z \in C_\varepsilon(L))$$

and $u := Tw$, $v := Tz$. Then by definition we have

$$\Big(1 + i\frac{\eta}{n_0}\Big)u + \frac{1}{n_0}(A + B_\varepsilon)u = g, \quad \Big(1 + i\frac{\eta}{n_0}\Big)v + \frac{1}{n_0}(A + B_\varepsilon)v = h. \tag{2.13}$$

It is easy to see that (2.12) is a consequence of the inequality

$$\|u - v\| \le (1 - (\tilde{C}_2/n_0)L^\theta)^{-1}\|g - h\|. \tag{2.14}$$

Making the difference of two equations in (2.13), we have

$$\Big(1 + i\frac{\eta}{n_0}\Big)(u - v) + \frac{1}{n_0}(Au - Av) + \frac{1}{n_0}(B_\varepsilon u - B_\varepsilon v) = g - h.$$

Taking the real part of the inner product of this equation with $u - v$, we see that

$$\begin{aligned} &\|u - v\|^2 + \frac{2\lambda}{n_0}\varphi(u - v) \\ &\le \frac{|\beta| - c_q^{-1}\kappa}{n_0}|\operatorname{Im}(\partial\psi_\varepsilon(u) - \partial\psi_\varepsilon(v), u - v)| + \|g - h\| \cdot \|u - v\|. \end{aligned}$$

Now we apply condition (**A5**) with $\rho := 2\lambda(|\beta| - c_q^{-1}\kappa)^{-1}$:

$$\begin{aligned} &\|u - v\|^2 - \|g - h\| \cdot \|u - v\| \\ &\quad \le \frac{|\beta| - c_q^{-1}\kappa}{n_0}\{|\operatorname{Im}(\partial\psi_\varepsilon(u) - \partial\psi_\varepsilon(v), u - v)| - \rho\,\varphi(u - v)\} \\ &\quad \le \frac{\tilde{C}_2}{n_0}\left(\frac{\psi(J_\varepsilon u) + \psi(J_\varepsilon v)}{2}\right)^\theta \|u - v\|^2. \end{aligned}$$

Since $\psi(J_\varepsilon w) \le \psi_\varepsilon(w) \le L$ for $w \in C_\varepsilon(L)$ (see (2.2)), we obtain (2.14). □

Now, taking account of conditions (**A4**) and (**A6**), the solvability of $(\mathbf{AStP})_\varepsilon$ is summarized as follows:

Proposition 2.12. *Let $f \in C_\varepsilon(L)$ and $\lambda^{-1}|\alpha| \le c_q^{-1}$. Assume that conditions* (**A1**)–(**A6**) *are satisfied. Then for every $\zeta \in \mathbb{C}$ with*

$$\xi := \operatorname{Re}\zeta \ge 1 + \tilde{C}_2 L^\theta$$

there exists a unique solution $u_\varepsilon = u_\varepsilon(\zeta) \in D(S) \cap C_\varepsilon(L)$ *to* $(\mathbf{AStP})_\varepsilon$*, with the following estimates*:

(a) $\|u_\varepsilon(\zeta)\| \le \xi^{-1}\|f\|, \quad 2\lambda\varphi(u_\varepsilon(\zeta)) + q\kappa\psi(u_\varepsilon(\zeta)) \le \xi^{-1}\|f\|^2$;

(b) $\psi_\varepsilon(u_\varepsilon(\zeta)) \le \psi_\varepsilon(f) \le L$.

Assume further that $f \in C_\varepsilon(L) \cap B_M = \{f \in X; \|f\| \le M, \psi_\varepsilon(f) \le L\}$. *Then* $u_\varepsilon(\zeta) \in B_M$ *and for* $\zeta \in \mathbb{C}$ *with* $\operatorname{Re}\zeta \ge 1 + \{\tilde{C}_2 L^\theta \vee \tilde{C}_3(M^2/(q\kappa))^\theta\}$ *one has additional estimates*:

(c) $\varphi(u_\varepsilon(\zeta)) + \dfrac{\lambda}{4}\|Su_\varepsilon(\zeta)\|^2 \le \dfrac{M^2}{\lambda}$;

(d) $\|\partial\psi_\varepsilon(u_\varepsilon(\zeta))\| \le \|\partial\psi(u_\varepsilon(\zeta))\| \le M\, C_1\Big(1 + \dfrac{2}{\lambda}\Big)$.

Furthermore, let v_ε *be a solution to* $(\mathbf{AStP})_\varepsilon$ *with right-hand side* $g \in C_\varepsilon(L) \cap B_M$. *Then for* $\zeta \in \mathbb{C}$ *with* $\operatorname{Re}\zeta \ge 1 + \{\tilde{C}_2 L^\theta \vee \tilde{C}_3(M^2/(q\kappa))^\theta\}$

$$\|u_\varepsilon(\zeta) - v_\varepsilon(\zeta)\| \le \left\{\xi - \tilde{C}_2\left(\frac{M^2}{q\kappa}\right)^\theta\right\}^{-1}\|f - g\|. \tag{2.15}$$

Proof. First we note that $D(A) = D(S) \subset D(\partial\psi)$ because of condition (**A4**). In view of Proposition 2.7 it remains to prove the estimates (a)–(d).

(a) Compute the real part after multiplying equation $(\mathbf{AStP})_\varepsilon$ by u_ε:

$$(\operatorname{Re}\zeta)\|u_\varepsilon\|^2 + 2\lambda\varphi(u_\varepsilon) + q\kappa\psi(u_\varepsilon) = \operatorname{Re}(f, u_\varepsilon) \le \|f\| \cdot \|u_\varepsilon\|.$$

(b) We compute the real part of the inner product of $(\mathbf{AStP})_\varepsilon$ with $\partial\psi_\varepsilon(u_\varepsilon)$:

$$\begin{aligned}&(\operatorname{Re}\zeta)\operatorname{Re}(u_\varepsilon, \partial\psi_\varepsilon(u_\varepsilon)) + \operatorname{Re}\{(\lambda + i\alpha)(Su_\varepsilon, \partial\psi_\varepsilon(u_\varepsilon))\}\\ &\quad = \operatorname{Re}(f - u_\varepsilon, \partial\psi_\varepsilon(u_\varepsilon)) + \operatorname{Re}(u_\varepsilon, \partial\psi_\varepsilon(u_\varepsilon)).\end{aligned}$$

Then by definition of subdifferential and Lemma 2.8 (c) with $v := u_\varepsilon$ we obtain

$$\psi_\varepsilon(u_\varepsilon) + (\operatorname{Re}\zeta - 1)q\,\psi(J_\varepsilon u_\varepsilon) + (c_q^{-1}\lambda - |\alpha|)|\operatorname{Im}(Su_\varepsilon, \partial\psi_\varepsilon(u_\varepsilon))| \le \psi_\varepsilon(f) \le L.$$

This shows under the condition $|\alpha| \le c_q^{-1}\lambda$ that $u_\varepsilon = u_\varepsilon(\zeta) \in C_\varepsilon(L)$ if $f \in C_\varepsilon(L)$.

(c) Compute the real part of the inner product of $(\mathbf{AStP})_\varepsilon$ with Su_ε:

$$\begin{aligned}&2(\operatorname{Re}\zeta)\varphi(u_\varepsilon) + \lambda\|Su_\varepsilon\|^2 + \operatorname{Re}\left\{i(\beta - \delta c_q^{-1}\kappa)(\partial\psi_\varepsilon(u_\varepsilon), Su_\varepsilon)\right\}\\ &\quad = \operatorname{Re}(f, Su_\varepsilon) \le \|f\| \cdot \|Su_\varepsilon\|;\end{aligned}$$

note that $\operatorname{Re}\{(\kappa + i\delta c_q^{-1}\kappa)\}(\partial\psi(u_\varepsilon), Su_\varepsilon)\} \ge 0$ by virtue of condition (**A3**) with $\partial\psi_\varepsilon$ replaced with $\partial\psi$ (by letting $\varepsilon \downarrow 0$). Applying (**A6**) with $\rho := (\lambda/4)(|\beta| - c_q^{-1}\kappa)^{-1}$, we see from (a) that

$$
\begin{aligned}
&2(\operatorname{Re}\zeta)\varphi(u_\varepsilon) + (\lambda/2)\|Su_\varepsilon\|^2 \\
&\quad \le \|f\|\cdot\|Su_\varepsilon\| + 2(|\beta| - c_q^{-1}\kappa)\bigl\{|\operatorname{Im}(Su_\varepsilon, \partial\psi_\varepsilon(u_\varepsilon))| - \rho\|Su_\varepsilon\|^2\bigr\}, \\
&\quad \le \|f\|\cdot\|Su_\varepsilon\| + 2\tilde{C}_3\psi(u_\varepsilon)^\theta\varphi(u_\varepsilon) \\
&\quad \le \|f\|\cdot\|Su_\varepsilon\| + 2\tilde{C}_3\left(\frac{\|f\|^2}{q\kappa}\right)^\theta \varphi(u_\varepsilon).
\end{aligned}
$$

Since $\operatorname{Re}\zeta - \tilde{C}_3(M^2/q\kappa)^\theta \ge 1$ and $\|f\| \le M$, we obtain

$$
\begin{aligned}
2\varphi(u_\varepsilon) + (\lambda/2)\|Su_\varepsilon\|^2 &\le 2\bigl(\operatorname{Re}\zeta - \tilde{C}_3(M^2/q\kappa)^\theta\bigr)\varphi(u_\varepsilon) + (\lambda/2)\|Su_\varepsilon\|^2 \\
&\le M\|Su_\varepsilon\| \le (2/\lambda)M^2.
\end{aligned}
$$

(d) is a consequence of (c) and condition (**A4**).

Finally, it remains to prove (2.15). Making the difference of two equations $(\mathbf{AStP})_\varepsilon$ with right-hand sides $f, g \in C_\varepsilon(L) \cap B_M$, we have

$$
\begin{aligned}
&(\operatorname{Re}\zeta)\|u_\varepsilon - v_\varepsilon\|^2 + 2\lambda\varphi(u_\varepsilon - v_\varepsilon) \\
&\le (|\beta| - c_q^{-1}\kappa)|\operatorname{Im}(\partial\psi_\varepsilon(u_\varepsilon) - \partial\psi_\varepsilon(v_\varepsilon), u_\varepsilon - v_\varepsilon)| + \|f-g\|\cdot\|u_\varepsilon - v_\varepsilon\|.
\end{aligned}
$$

Applying condition $(\mathbf{A5})'$ with $\rho := 2\lambda(|\beta| - c_q^{-1}\kappa)^{-1}$, we see from (a) that

$$
\begin{aligned}
&(\operatorname{Re}\zeta)\|u_\varepsilon - v_\varepsilon\|^2 - \|f-g\|\cdot\|u_\varepsilon - v_\varepsilon\| \\
&\quad \le (|\beta| - c_q^{-1}\kappa)\bigl\{|\operatorname{Im}(\partial\psi_\varepsilon(u_\varepsilon) - \partial\psi_\varepsilon(v_\varepsilon), u_\varepsilon - v_\varepsilon)| - \rho\,\varphi(u_\varepsilon - v_\varepsilon)\bigr\} \\
&\quad \le \tilde{C}_2\left(\frac{\psi(u_\varepsilon) + \psi(v_\varepsilon)}{2}\right)^\theta \|u_\varepsilon - v_\varepsilon\|^2 \\
&\quad \le \tilde{C}_2(M^2/q\kappa)^\theta\|u_\varepsilon - v_\varepsilon\|^2.
\end{aligned}
$$

This proves (2.15). □

For $M > 0$ fix a *new* closed convex subset in X:

$$
C(M) := \{w \in X; \|w\| \le M,\ \psi(w) \le M^2/(q\kappa)\}. \tag{2.16}
$$

Choosing $C_\varepsilon(L)$ with $L := M^2/(q\kappa)$, we see from (2.2) that

$$
C(M) \subset C_\varepsilon(M^2/(q\kappa)) \cap B_M \ \ \forall\, \varepsilon > 0.
$$

Lemma 2.13. *Let $f \in C(M)$ and $|\alpha| \le c_q^{-1}\lambda$. Assume that conditions* (**A1**)−(**A7**) *are satisfied. Let $\{u_\nu(\zeta)\}_{\nu>0}$ be a family of solutions to $(\mathbf{AStP})_\nu$ as constructed in* Proposition 2.12 *with $f \in C(M)$. Then for $\zeta \in \mathbb{C}$ with*

$$
\operatorname{Re}\zeta \ge 1 + (\tilde{C}_2 \vee \tilde{C}_3)(M^2/q\kappa)^\theta
$$

the family $\{u_\nu\}$ satisfies Cauchy's condition in X :

$$
\left\{\operatorname{Re}\zeta - \tilde{C}_2\left(\frac{M^2}{q\kappa}\right)^\theta\right\}\|u_\nu - u_\mu\|^2 \le M^2C_1^2\tilde{C}_4\left(1 + \frac{2}{\lambda}\right)^2|\nu - \mu|. \tag{2.17}
$$

Proof. Let $f \in C(M)$. Then there is a unique solution u_ν to $(\mathbf{AStP})_\nu$ such that

$$\psi(u_\nu(\zeta)) \le \frac{1}{q\kappa \operatorname{Re}\zeta}\|f\|^2 \le \frac{M^2}{q\kappa} \quad \text{for } \operatorname{Re}\zeta \ge 1 + \tilde{C}_2\left(\frac{M^2}{q\kappa}\right)^\theta, \tag{2.18}$$

$$\|\partial\psi(u_\nu(\zeta))\| \le M\, C_1\Big(1+\frac{2}{\lambda}\Big) \quad \text{for } \operatorname{Re}\zeta \ge 1 + (\tilde{C}_2 \vee \tilde{C}_3)\left(\frac{M^2}{q\kappa}\right)^\theta \tag{2.19}$$

(see Proposition 2.12 (a) and (d)). Making the difference of equations $(\mathbf{AStP})_\varepsilon$ with $\varepsilon = \nu,\ \mu$, we have

$$(\operatorname{Re}\zeta)\|u_\nu - u_\mu\|^2 + 2\lambda\varphi(u_\nu - u_\mu) \le (|\beta| - c_q^{-1}\kappa)|\operatorname{Im}(\partial\psi_\nu(u_\nu) - \partial\psi_\mu(u_\mu), u_\nu - u_\mu)|.$$

Setting $\rho := 2\lambda(|\beta| - c_q^{-1}\kappa)^{-1}$, we have

$$\begin{aligned}
&(\operatorname{Re}\zeta)\|u_\nu - u_\mu\|^2\\
&\quad\le (|\beta| - c_q^{-1}\kappa)\big\{|\operatorname{Im}(\partial\psi_\nu(u_\nu) - \partial\psi_\nu(u_\mu), u_\nu - u_\mu)| - \rho\varphi(u_\nu - u_\mu)\big\}\\
&\qquad + (|\beta| - c_q^{-1}\kappa)|\operatorname{Im}(\partial\psi_\nu(u_\mu) - \partial\psi_\mu(u_\mu), u_\nu)|;
\end{aligned}$$

note that $\operatorname{Im}(\partial\psi_\varepsilon(u_\mu), u_\mu) = 0$ for $\varepsilon = \nu,\ \mu$. Applying conditions $(\mathbf{A5})'$ and $(\mathbf{A7})$ to the first and second terms on the right-hand side, we see from (2.18) and (2.19) that

$$\begin{aligned}
&(\operatorname{Re}\zeta)\|u_\nu - u_\mu\|^2\\
&\le \tilde{C}_2\bigg(\frac{\psi(u_\nu)+\psi(u_\mu)}{2}\bigg)^\theta \|u_\nu - u_\mu\|^2 + \tilde{C}_4|\nu-\mu|\left(\sigma\|\partial\psi(u_\mu)\|^2 + \tau\|\partial\psi(u_\nu)\|^2\right)\\
&\le \tilde{C}_2(M^2/q\kappa)^\theta\|u_\nu - u_\mu\|^2 + M^2C_1^2\tilde{C}_4(1+2/\lambda)^2|\nu-\mu|.
\end{aligned}$$

This proves (2.17). □

Now we can complete

Proof of Theorem 2.3. Part (**II**). Put $u(\zeta) := \lim_{\varepsilon\to 0} u_\varepsilon(\zeta)$. Then it remains to prove that the limit $u(\zeta)$ is a unique strong solution to (**AStP**):

$$(\xi + i\eta)u(\zeta) + (\lambda + i\alpha)Su(\zeta) + (\kappa + i\beta)\partial\psi(u(\zeta)) = f.$$

First we note that $u(\zeta) = \lim_{\varepsilon\downarrow 0} J_\varepsilon u_\varepsilon(\zeta)$. In fact, by the definition of $\partial\psi_\varepsilon$ we have

$$\|u(\zeta) - J_\varepsilon u_\varepsilon(\zeta)\| \le \|u(\zeta) - u_\varepsilon(\zeta)\| + \varepsilon\|\partial\psi_\varepsilon(u_\varepsilon(\zeta))\| \quad \forall\, \varepsilon > 0.$$

We need the boundedness of $\{\partial\psi_\varepsilon(u_\varepsilon(\zeta))\}$ to conclude the convergence (see Proposition 2.12 (d)). Since S, $\partial\psi$ are maximal monotone in X, we see from Proposition 2.12 (c), (d) that $u(\zeta) \in D(S) \subset D(\partial\psi)$ and

$$Su(\zeta) = \operatorname*{w-lim}_{\varepsilon\downarrow 0} Su_\varepsilon(\zeta), \quad \partial\psi(u(\zeta)) = \operatorname*{w-lim}_{\varepsilon\downarrow 0}\partial\psi(u_\varepsilon(\zeta)) = \operatorname*{w-lim}_{\varepsilon\downarrow 0}\partial\psi_\varepsilon(u_\varepsilon(\zeta))$$

(see Kato [7, Lemma 2.5]). Now, letting ε tend to zero in $(\mathbf{AStP})_\varepsilon$ and (2.15), we see that $u(\zeta)$ is a strong solution to (**AStP**) satisfying (2.6). Obviously, the uniqueness of solutions to (**AStP**) is guaranteed by the estimate (2.6) of the continuous dependence on right-hand sides.

Finally, estimate (a)–(c) except that for $\partial\psi(u(\zeta))$ follow from Proposition 2.12 (a) and (c). In fact, we can use the lower semi-continuity of φ, ψ and calculus inequality:

$$2\lambda \varliminf_{\varepsilon\downarrow 0} \varphi(u_\varepsilon) + q\kappa \varliminf_{\varepsilon\downarrow 0} \psi(u_\varepsilon) \le \varliminf_{\varepsilon\downarrow 0} (2\lambda\varphi(u_\varepsilon) + q\kappa\psi(u_\varepsilon)).$$

On the other hand, since $\mathrm{Re}\{(\lambda + i\alpha)(Su(\zeta), \partial\psi(u(\zeta)))\} \ge 0$, we obtain

$$q\xi\psi(u(\zeta)) + \kappa\|\partial\psi(u(\zeta))\|^2 \le \mathrm{Re}\,(f, \partial\psi(u(\zeta)))$$

from which we can conclude the second half of (c). □

3. Proof of Theorem 1.2

Theorem 1.2 is proved by applying Theorem 2.1 to (**StCGL**). Namely, it is applied to S and ψ as defined in Section 1. The verification of conditions (**A1**)–(**A3**) is done in [10, Section 6], while conditions (**A4**) and (**A6**) have been verified in [11, Section 3]. Actually, $\psi(u)$ in condition (**A6**) is replaced with $\psi(J_\varepsilon u)$ in [11]. Therefore condition (**A6**) is weaker than that in [11]. On the other hand, (**A7**) is a new condition introduced and verified in [12] which guarantees, together with condition (**A5**), the convergence of the family of approximate solutions.

Here we present a proof of condition (**A5**) because what have been verified in [11] and [12] are respectively conditions (**A5**)′ and (**A5**)″ (see Remark 2.1).

Lemma 3.1. *Let $a, b \in \mathbb{C}$ and $q \ge 2$. For $\varepsilon > 0$ define z_ε, w_ε as*

$$z_\varepsilon + \varepsilon|z_\varepsilon|^{q-2}z_\varepsilon = a, \quad w_\varepsilon + \varepsilon|w_\varepsilon|^{q-2}w_\varepsilon = b.$$

Then one has

$$\big|\mathrm{Im}\{\overline{(a-b)}(|z_\varepsilon|^{q-2}z_\varepsilon - |w_\varepsilon|^{q-2}w_\varepsilon)\}\big| \le \frac{q-2}{2}|a-b|^2\Big(\frac{1}{2}|z_\varepsilon|^q + \frac{1}{2}|w_\varepsilon|^q\Big)^{(q-2)/q}. \tag{3.1}$$

Proof. First, the accretivity of the mapping $z \mapsto |z|^{q-2}z$ implies that

$$|z_\varepsilon - w_\varepsilon| \le |a-b|. \tag{3.2}$$

Next, put $F(z,w) := \overline{(z-w)}(tz + (1-t)w)$. Then the fundamental theorem of calculus yields

$$\begin{aligned}
&|z|^{q-2}z - |w|^{q-2}w \\
&\quad = \int_0^1 \frac{d}{dt}\big\{|tz + (1-t)w|^{q-2}(tz + (1-t)w)\big\}\,dt \\
&\quad = (z-w)\int_0^1 |tz + (1-t)w|^{q-2}\,dt \\
&\qquad + (q-2)\int_0^1 |tz + (1-t)w|^{q-4}(tz + (1-t)w)\mathrm{Re}\big\{F(z,w)\big\}\,dt.
\end{aligned}$$

Setting $z := z_\varepsilon$ and $w := w_\varepsilon$, we have

$$\mathrm{Im}\{\overline{(a-b)}(|z_\varepsilon|^{q-2}z_\varepsilon - |w_\varepsilon|^{q-2}w_\varepsilon)\} = \mathrm{Im}\{\overline{(z_\varepsilon - w_\varepsilon)}(|z_\varepsilon|^{q-2}z_\varepsilon - |w_\varepsilon|^{q-2}w_\varepsilon)\}$$
$$= (q-2)\int_0^1 |tz_\varepsilon + (1-t)w_\varepsilon|^{q-4}\,\mathrm{Re}\{F(z_\varepsilon, w_\varepsilon)\}\mathrm{Im}\{F(z_\varepsilon, w_\varepsilon)\}\,dt.$$

Since $2|\mathrm{Re}\,z|\cdot|\mathrm{Im}\,z| \le |z|^2\ \forall\ z \in \mathbb{C}$, it follows from (3.2) and Hölder's inequality that

$$\begin{aligned}&\big|\mathrm{Im}\{\overline{(a-b)}(|z_\varepsilon|^{q-2}z_\varepsilon - |w_\varepsilon|^{q-2}w_\varepsilon)\}\big| \\ &\quad\le \frac{q-2}{2}|a-b|^2\int_0^1 |tz_\varepsilon + (1-t)w_\varepsilon|^{q-2}\,dt \\ &\quad\le \frac{q-2}{2}|a-b|^2\left(\int_0^1 |tz_\varepsilon + (1-t)w_\varepsilon|^{q}\,dt\right)^{(q-2)/q}.\end{aligned}$$

The convexity of the function: $z \mapsto |z|^q$ applies to give the desired inequality (3.1). □

Remark 3.2. Since $|z_\varepsilon| \le |a|$, $|w_\varepsilon| \le |b|$, $|z_\varepsilon|$ and $|w_\varepsilon|$ on the right-hand side of (3.1) may be replaced with $|a|$ and $|b|$, respectively. This form of (3.1) has been proved in [12].

Let ψ be as defined by (1.1). Then by virtue of Lemma 3.1 we have

Lemma 3.3. *Let u, $v \in L^2(\Omega)$ and $q \ge 2$. Then for $\varepsilon > 0$,*

$$|\mathrm{Im}(\partial\psi_\varepsilon(u) - \partial\psi_\varepsilon(v), u - v)_{L^2}| \le \frac{q-2}{2}\|u-v\|_{L^q}^2\left(\frac{q}{2}\psi(J_\varepsilon u) + \frac{q}{2}\psi(J_\varepsilon v)\right)^{(q-2)/q}, \tag{3.3}$$

where $\partial\psi_\varepsilon$ is the Yosida approximation of $\partial\psi$ and $J_\varepsilon = (1 + \varepsilon\partial\psi)^{-1}$.

In fact, (3.1) with $a = u(x)$, $b = v(x)$ and $z_\varepsilon = (J_\varepsilon u)(x)$, $w_\varepsilon = (J_\varepsilon v)(x)$ gives us

$$\begin{aligned}&\big|\mathrm{Im}\{\overline{[u(x) - v(x)]}(|u_\varepsilon(x)|^{q-2}u_\varepsilon(x) - |v_\varepsilon(x)|^{q-2}v_\varepsilon(x))\}\big| \\ &\quad\le \frac{q-2}{2}|u(x) - v(x)|^2\left(\frac{1}{2}|(J_\varepsilon u)(x)|^q + \frac{1}{2}|(J_\varepsilon v)(x)|^q\right)^{(q-2)/q}.\end{aligned}$$

Integrating this inequality over Ω and applying Hölder's inequality we can obtain (3.3).

To derive condition (**A5**) from (3.3) we shall use the Gagliardo-Nirenberg inequality:

$$\|w\|_{L^q} \le C_0\|w\|_{L^2}^{1-a}\|\nabla w\|_{L^2}^a\ \ \forall\ w \in H^1(\Omega),\ \text{ with }\ a := \frac{N}{2} - \frac{N}{q} \in [0,1]; \tag{3.4}$$

note that $a \in [0, 1/2]$ if $N = 1$ (see Tanabe [14, Section 3.4] or Giga-Giga [5, Chapter 6]). However, we have to avoid the case of $a = 1$. In fact, $a = 1$ corresponds to the critical power $q = 2^*$ in the sense of Sobolev embedding exponent. Thus we impose the subcritical condition on q:

$$\begin{cases} 2 \le q < 2^* := \dfrac{2N}{N-2} & (N \ge 3), \\ 2 \le q < \infty & (N = 1, 2). \end{cases} \tag{3.5}$$

Incidentally, it is easy to prove (3.4) if $N \ge 3$ (and $N = 1$). In fact, let $2 \le q \le 2^*$. Then it follows from Hölder's inequality that

$$\|w\|_{L^q} \le \|w\|_{L^2}^{1-a} \|w\|_{L^{2^*}}^a \quad \forall\, w \in L^2(\Omega) \cap L^{2^*}(\Omega), \quad \frac{1}{q} = \frac{1-a}{2} + \frac{a}{2^*}.$$

Therefore (3.4) is a direct consequence of the Sobolev inequality

$$\|w\|_{L^{2^*}} \le C' \|\nabla w\|_{L^2} \quad \forall\, w \in H^1(\Omega), \quad \frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}$$

(use $2^* = \infty$ and $\|w\|_{L^\infty}^2 \le C'\|w\|_{L^2}\|\nabla w\|_{L^2}$ if $N = 1$). Now, setting

$$\varphi(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 \text{ for } u \in D(\varphi) := H_0^1(\Omega),$$

we have

Lemma 3.4. *Let q satisfy* (3.5)*. Then for any $\rho > 0$ there exists a constant $C_2 = C_2(\rho)$ such that for $u, v \in D(\varphi)$ and $\varepsilon > 0$,*

$$|\mathrm{Im}(\partial\psi_\varepsilon(u) - \partial\psi_\varepsilon(v), u - v)_{L^2}| \le \rho\,\varphi(u - v) + C_2\left(\frac{\psi(J_\varepsilon u) + \psi(J_\varepsilon v)}{2}\right)^\theta \|u - v\|_{L^2}^2,$$

where $\theta := \dfrac{q-2}{(1-a)q}$ coincides with $f_N(q)$ as in Section 1.

In fact, using the Gagliardo-Nirenberg inequality and then Young's inequality, we see from (3.3) that for any $\rho > 0$ there exists $C_2 = C_2(\rho) > 0$ such that

$$\begin{aligned} &|\mathrm{Im}(\partial\psi_\varepsilon(u) - \partial\psi_\varepsilon(v), u - v)_{L^2}| \\ &\qquad \le \rho\,\|\nabla(u - v)\|_{L^2}^2 + C_2\Big(\frac{q}{2}\psi(J_\varepsilon u) + \frac{q}{2}\psi(J_\varepsilon v)\Big)^\theta \|u - v\|_{L^2}^2; \end{aligned}$$

note that $\theta \in [0, \infty) \Leftrightarrow a \in [0, 1) \Leftrightarrow q \in [2, 2^*)$.

Remark 3.5. We have already noted that $\theta \in [0, 1] \Leftrightarrow q \in [2, 2 + 4/N]$.

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Operator-valued Symbols for Elliptic and Parabolic Problems on Wedges

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Dedicated to Philippe Clément

Abstract. We study evolution problems of the type $e^{sx}\partial_t u + h(\partial_x)u = f$ where h is a holomorphic function on a vertical strip around the imaginary axis, and $s > 0$. If P is a second-order polynomial we give a complete characterization of the spectrum of the parameter-dependent operator $\lambda e^{sx} + P(\partial_x)$ in $L_p(\mathbb{R})$. We show the surprising result that the spectrum is independent of λ whenever $|\arg\lambda| < \pi$. Moreover, we also characterize the spectrum of $\partial_t e^{sx} + P(\partial_x)$, and we show that this operator admits a bounded $\mathcal{H}^\infty$-calculus. Finally, we describe applications to free boundary problems with moving contact lines, and we study the diffusion equation in an angle or a wedge domain with dynamic boundary conditions. Our approach relies on the $\mathcal{H}^\infty$-calculus for sectorial operators, the concept of $\mathcal{R}$-boundedness, and recent results for the sum of non-commuting operators.

1. Introduction

In recent years the $\mathcal{H}^\infty$-calculus for sectorial operators in Banach spaces, the concept of $\mathcal{R}$-boundedness, and the method of operator sums have become important tools for proving existence and optimal regularity results for solutions of partial differential and integro-differential equations, as well as for abstract evolutionary problems. We mention here only the references [7, 8, 15, 16, 17] which document some recent work by the authors. In the current paper we apply these techniques to the study of certain operator-valued symbols which arise from elliptic or parabolic equations on angles or wedges with dynamic boundary conditions. Another main objective of this paper is to develop tools for the study of free boundary problems with moving contact lines.

To describe the class of symbols we have in mind, let us first consider the case of dynamic boundary conditions. It is shown in Section 5 that the boundary symbol for the Laplace equation $\Delta u = 0$ on an angle $G = \{(r\cos\phi, r\sin\phi);\ r > 0,\ \phi \in (0,\alpha)\}$ in $\mathbb{R}^2$ with Dirichlet condition $u = 0$ on $\phi = \alpha$ and dynamic boundary condition $\partial_t u + \partial_\nu u = g$ on $\phi = 0$ is given by

$$\partial_t e^x + \psi_0(-(\partial_x + \beta)^2), \quad \psi_0(z) = \sqrt{z}\coth(\alpha\sqrt{z}), \quad z \in \mathbb{C}.$$

Here $\beta \in \mathbb{R}$ is a number whose meaning is explained in Section 5. Similarly, if one considers the one-phase quasi-stationary Stefan problem with surface tension (also sometimes called the Mullins-Sekerka problem) in two dimensions with boundary intersection and prescribed contact angle $\alpha \in (0,\pi]$, one is led to the boundary symbol

$$\partial_t e^{3x} - \psi_1\big(\partial_x + \beta)^2\big)(\partial_x + \beta + 1)(\partial_x + \beta + 2),$$

where this time $\psi_1(z) = \sqrt{z}\tanh(\alpha\sqrt{z})$. The free boundary problem for the stationary Stokes equations with boundary contact and prescribed contact angle in two dimensions leads to

$$\partial_t e^x + \psi(\partial_x + \beta),$$

where

$$\psi(z) = (1+z)\frac{\cos(2\alpha z) - \cos(2\alpha)}{\sin(2\alpha z) + z\sin(2\alpha)},$$

in the slip case and

$$\psi(z) = \frac{(1+z)}{4}\frac{\sin(2\alpha z) - z\sin(2\alpha)}{z^2\sin^2(\alpha) - \cos^2(\alpha z)}$$

in the non-slip case. This motivates the study of equations of the type

$$\partial_t e^{sx} + h(\partial_x) \tag{1.1}$$

and its parametric form

$$\lambda e^{sx} + h(\partial_x), \tag{1.2}$$

where $s > 0$, $\lambda \in \mathbb{C}$, and h is a function holomorphic on a vertical strip around the imaginary axis.

In higher dimensions, the boundary symbol for the Laplace equation in a wedge with dynamic boundary condition is given by

$$\partial_t e^x + \psi_0\big(-\Delta_y e^{2x} - (\partial_x + \beta)^2\big),$$

where Δ_y means the Laplacian in the variables y tangential to the edge, whereas that of the quasi-stationary Stefan problem becomes

$$\partial_t e^{3x} + \psi_1\big(-\Delta_y e^{2x} - (\partial_x + \beta)^2\big)[-\Delta_y e^{2x} - (\partial_x + \beta + 1)(\partial_x + \beta + 2)].$$

Similarly,

$$\partial_t e^x + \psi_0\big((\partial_t - \Delta_y)e^{2x} - (\partial_x + \beta)^2\big)$$

represents the boundary symbol for the diffusion equation in a wedge with dynamic boundary condition. Clearly, these symbols are considerably more complicated than in the two-dimensional case. They also show the importance of the special case $h = P$ where P is a second-order polynomial. In the current paper we concentrate on this case.

It is our ultimate goal to identify function spaces such that the operators defined by the symbols of type (1.1) and (1.2) become topological isomorphisms between these spaces, i.e., to obtain optimal solvability results. We intend to do this in the framework of L_p-spaces. Our main tools are very recent results on sums of sectorial operators, their $\mathcal{H}^\infty$-calculi, and $\mathcal{R}$-boundedness of associated operator families.

Once this goal is achieved, one can go on to study symbols of higher-dimensional (or time-dependent) problems. We will do this here only for those arising from the problems with dynamic boundary conditions. The symbols for the Mullins-Sekerka problem in higher dimensions, for the Stefan problem with surface tension, and for the non-steady Stokes problem with free boundary will be the subject of future work.

Observe that symbols of type (1.1) and (1.2) are highly degenerate, due to the presence of the exponentials. They are not directly accessible by standard methods for pseudo-differential operators. Moreover, the basic ingredients of these symbols, namely e^x and ∂_x, do not commute and so the functional calculus in two variables does not apply. Still, there is a close relation between these operators. In fact, e^{sx} is an eigenfunction of ∂_x with eigenvalue s, or to put it in a different way, the commutator between e^{sx} and ∂_x is se^{sx}. It is this relation we base our approach on. It allows us to apply abstract results on sums of non-commuting operators.

The plan for this paper is as follows. In Section 2 we introduce the necessary notation and state some abstract results needed in the sequel. In Sections 3 and 4 we deal with the important special case where h is a second-order polynomial P. By means of an explicit representation of the resolvent we derive a complete characterization of the spectrum of the operators

$$\lambda e^{sx} + P(\partial_x) \quad \text{in} \quad L_p(\mathbb{R})$$

and

$$\partial_t e^{sx} + P(\partial_x) \quad \text{in} \quad L_p(J \times \mathbb{R}),$$

where $J = (0, T)$. We conclude the paper with applications to the Laplace equation and the diffusion equation in an angle or a wedge with dynamic boundary condition. By means of our results we obtain the complete solvability behavior of this problem in appropriately weighted L_p-spaces.

We mention here that the Laplace and the diffusion equation with dynamic boundary conditions in domains with edges were previously studied in the framework of weighted L_2-spaces by Frolova and Solonnikov, see [10, 18] and the references contained therein.

2. Preliminaries

In the following, $X = (X, |\cdot|)$ always denotes a Banach space with norm $|\cdot|$, and $\mathcal{B}(X)$ stands for the space of all bounded linear operators on X, where we will again use the notation $|\cdot|$ for the norm in $\mathcal{B}(X)$. If A is a linear operator on X, then $\mathsf{D}(A)$, $\mathsf{R}(A)$, $\mathsf{N}(A)$ denote the domain, the range, and the kernel of A, whereas $\rho(A)$, $\sigma(A)$ stand for the resolvent set, and the spectrum of A, respectively. An operator A is called *sectorial* if

- $\mathsf{D}(A)$ and $\mathsf{R}(A)$ are dense in X,
- $(-\infty, 0) \subset \rho(A)$ and $|t(t+A)^{-1}| \le M$ for $t > 0$.

The class of all sectorial operators is denoted by $\mathcal{S}(X)$. If A is sectorial, then it is closed, and it follows from the ergodic theorem that $\mathsf{N}(A) = 0$. Moreover, by a Neumann series argument one obtains that $\rho(-A)$ contains a sector

$$\Sigma_\phi := \{z \in \mathbb{C} : \ z \neq 0, \ |\arg(z)| < \phi\}.$$

Consequently, it is meaningful to define the *spectral angle* ϕ_A of A by means of

$$\phi_A := \inf\{\phi > 0 : \rho(-A) \supset \Sigma_{\pi-\phi}, \ M_{\pi-\phi} < \infty\},$$

where $M_\theta := \sup\{|\lambda(\lambda+A)^{-1}| : \ \lambda \in \Sigma_\theta\}$. Obviously we have

$$\pi > \phi_A \ge \arg(\sigma(A)) := \sup\{|\arg(\lambda)| : \ \lambda \neq 0, \ \lambda \in \sigma(A)\}.$$

If A is sectorial, the functional calculus of Dunford given by

$$\Phi_A(f) := f(A) := \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - A)^{-1} d\lambda$$

is a well-defined algebra homomorphism $\Phi_A : \mathcal{H}_0(\Sigma_\phi) \to \mathcal{B}(X)$, where $\mathcal{H}_0(\Sigma_\phi)$ denotes the set of all functions $f : \Sigma_\phi \to \mathbb{C}$ that are holomorphic and that satisfy the condition

$$\sup_{\lambda \in \Sigma_\phi} (|\lambda^{-\varepsilon} f(\lambda)| + |\lambda^\varepsilon f(\lambda)|) < \infty \text{ for some } \varepsilon > 0 \text{ and some } \phi > \phi_A.$$

Here Γ denotes a contour $\Gamma = e^{i\theta}(\infty, 0] \cup e^{-i\theta}[0, \infty)$ with $\theta \in (\phi_A, \phi)$. A is said to admit an $\mathcal{H}^\infty$*-calculus* if there are numbers $\phi > \phi_A$ and $M > 0$ such that the estimate

$$|f(A)| \le M|f|_{\mathcal{H}^\infty(\Sigma_\phi)}, \quad f \in \mathcal{H}_0(\Sigma_\phi), \tag{2.1}$$

is valid. In this case, the Dunford calculus extends uniquely to $\mathcal{H}^\infty(\Sigma_\phi)$, see for instance [6] for more details. We denote the class of sectorial operators which admit an $\mathcal{H}^\infty$-calculus by $\mathcal{H}^\infty(X)$. The infimum ϕ_A^∞ of all angles ϕ such that (2.1) holds for some constant $M > 0$ is called the $\mathcal{H}^\infty$*-angle* of A.

Let $\mathcal{T} \subset \mathcal{B}(X)$ be an arbitrary set of bounded linear operators on X. Then $\mathcal{T}$ is called $\mathcal{R}$*-bounded* if there is a constant $M > 0$ such that the inequality

$$\mathbb{E}(|\sum_{i=1}^N \varepsilon_i T_i x_i|) \le M \mathbb{E}(|\sum_{i=1}^N \varepsilon_i x_i|) \tag{2.2}$$

is valid for every $N \in \mathbb{N}$, $T_i \in \mathcal{T}$, $x_i \in X$, and all independent, symmetric $\{\pm 1\}$-valued random variables ε_i on a probability space $(\Omega, \mathcal{A}, P)$ with expectation $\mathbb{E}$. The smallest constant M in (2.2) is called the $\mathcal{R}$*-bound* of $\mathcal{T}$ and is denoted by $\mathcal{R}(\mathcal{T})$. A sectorial operator A is called $\mathcal{R}$*-sectorial* if the set

$$\{\lambda(\lambda + A)^{-1} : \ \lambda \in \Sigma_{\pi-\phi}\} \text{ is } \mathcal{R}\text{-bounded for some } \phi \in (0, \pi).$$

The infimum ϕ_A^R of such angles ϕ is called the *$\mathcal{R}$-angle* of A. We denote the class of $\mathcal{R}$-sectorial operators by $\mathcal{RS}(X)$. The relation $\phi_A^R \geq \phi_A$ is clear. If X is a space of class $\mathcal{HT}$ and $A \in \mathcal{H}^\infty(X)$ then it follows from a result of Clément and Prüss [3] that $A \in \mathcal{RS}(X)$ with $\phi_A^R \leq \phi_A^\infty$.

Finally, an operator $A \in \mathcal{H}^\infty(X)$ is said to admit an *$\mathcal{R}$-bounded* $\mathcal{H}^\infty$-calculus if the set

$$\{f(A) : \ f \in \mathcal{H}^\infty(\Sigma_\phi), |f|_{\mathcal{H}^\infty(\Sigma_\phi)} \leq 1\}$$

is $\mathcal{R}$-bounded for some $\phi \in (0, \pi)$. Again, the infimum $\phi_A^{R\infty}$ of such ϕ is called the $\mathcal{RH}^\infty$-angle of A, and the class of such operators is denoted by $\mathcal{RH}^\infty(X)$.

If X enjoys the so-called *property* (α), see [2, Definition 3.11], then every operator $A \in \mathcal{H}^\infty(X)$ already has an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus, that is, $\mathcal{H}^\infty(X) = \mathcal{RH}^\infty(X)$ and $\phi_A^{R\infty} = \phi_A^\infty$, see [12, Theorem 5.3]. In particular, the L_p-spaces with $1 < p < \infty$ have property (α), see [2].

We refer to the monograph of Denk, Hieber, and Prüss [6] for further information and background material.

We now state a recent result on the existence of an operator-valued $\mathcal{H}^\infty$-calculus proved in [12], the general Kalton-Weis theorem.

Theorem 2.1. *Let X be a Banach space, $A \in \mathcal{H}^\infty(X)$, $F \in \mathcal{H}^\infty(\Sigma_\phi; \mathcal{B}(X))$ such that*

$$F(\lambda)(\mu - A)^{-1} = (\mu - A)^{-1} F(\lambda), \quad \mu \in \rho(A), \ \lambda \in \Sigma_\phi.$$

Suppose $\phi > \phi_A^\infty$ and $\mathcal{R}(F(\Sigma_\phi)) < \infty$. Then

$$F(A) \in \mathcal{B}(X) \quad \text{and} \quad |F(A)|_{\mathcal{B}(X)} \leq C_A \mathcal{R}(F(\Sigma_\phi))$$

where C_A denotes a constant depending on A but not on F.

Given two sectorial operators A and B we define

$$(A + B)x := Ax + Bx, \quad x \in \mathsf{D}(A + B) := \mathsf{D}(A) \cap \mathsf{D}(B).$$

A and B are said to *commute* if there are numbers $\lambda \in \rho(A)$ and $\mu \in \rho(B)$ such that

$$(\lambda - A)^{-1}(\mu - B)^{-1} = (\mu - B)^{-1}(\lambda - A)^{-1}.$$

In this case, the commutativity relation holds for all $\lambda \in \rho(A)$ and $\mu \in \rho(B)$.

In their famous paper [5] Da Prato and Grisvard proved the following result: suppose $A, B \in \mathcal{S}(X)$ commute and the *parabolicity condition* $\phi_A + \phi_B < \pi$ holds true. Then $A + B$ is closable and its closure $L := \overline{A + B}$ is again sectorial with spectral angle $\phi_L \leq \max\{\phi_A, \phi_B\}$.

The natural question in this context then is whether $A + B$ is already closed, i.e., if *maximal regularity* holds. There are many contributions to this question, see for instance the discussion in Prüss and Simonett [17].

Applied to the functions $F(z) = z(z+B)^{-1}$ and $F(z) = f(z+B)$, Theorem 2.1 implies the following result on sums of commuting operators: suppose $A \in \mathcal{H}^\infty(X)$ and $B \in \mathcal{RS}(X)$, A, B commute, and $\phi_A^\infty + \phi_B^R < \pi$. Then $A + B$ is closed and sectorial. If in addition $B \in \mathcal{RH}^\infty(X)$ and $\phi_A^\infty + \phi_B^{R\infty} < \pi$, then $A + B$ has an $\mathcal{H}^\infty$-calculus as well.

We now turn to the non-commuting case and we assume that A and B satisfy the Labbas-Terreni commutator condition, which reads as follows.

$$\left\{\begin{array}{l} 0 \in \rho(A). \text{ There are constants } c > 0,\ \ 0 \le \alpha < \beta < 1, \\ \psi_A > \phi_A,\ \psi_B > \phi_B,\ \psi_A + \psi_B < \pi, \\ \text{ such that for all } \lambda \in \Sigma_{\pi-\psi_A},\ \mu \in \Sigma_{\pi-\psi_B} \\ |A(\lambda + A)^{-1}[A^{-1}(\mu + B)^{-1} - (\mu + B)^{-1}A^{-1}]| \le c/(1 + |\lambda|)^{1-\alpha}|\mu|^{1+\beta}. \end{array}\right. \tag{2.3}$$

Assuming this condition we have the following generalization of the Kalton-Weis theorem on sums of operators to the non-commuting case proved recently by the authors in [17].

Theorem 2.2. *Suppose $A \in \mathcal{H}^\infty(X)$, $B \in \mathcal{RS}(X)$ and suppose that* (2.3) *holds for some angles $\psi_A > \phi_A^\infty$, $\psi_B > \phi_B^R$ with $\psi_A + \psi_B < \pi$.*
Then there is a number $\omega_0 \ge 0$ such that $\omega_0 + A + B$ is invertible and sectorial with angle $\phi_{\omega_0+A+B} \le \max\{\psi_A, \psi_B\}$. Moreover, if in addition $B \in \mathcal{RH}^\infty(X)$ and $\psi_B > \phi_B^{R\infty}$, then

$$\omega_0 + A + B \in \mathcal{H}^\infty(X) \quad \text{and} \quad \phi^\infty_{\omega_0+A+B} \le \max\{\psi_A, \psi_B\}.$$

This result will be one of the main tools for the theory developed below.

If A and B are sectorial operators on X, their product is defined by

$$(AB)x := ABx, \quad \mathsf{D}(AB) := \{x \in \mathsf{D}(B) :\ Bx \in \mathsf{D}(A)\}.$$

Closedness of the product is easier to obtain than for sums, as AB is closed as soon as A is invertible or B is bounded. On the other hand, it is in general not a simple task to prove that AB is again sectorial. The following assertive result is a consequence of the general Kalton-Weis theorem: suppose that A and B are sectorial commuting operators in X, that A is invertible, and suppose that $A \in \mathcal{H}^\infty(X)$, $B \in \mathcal{RS}(X)$ such that $\phi_A^\infty + \phi_B^R < \pi$. Then AB is sectorial with angle $\phi_{AB} \le \phi_A^\infty + \phi_B^R$. If moreover $B \in \mathcal{RH}^\infty(X)$ and $\phi_A^\infty + \phi_B^{R\infty} < \pi$, then

$$AB \in \mathcal{H}^\infty(X) \quad \text{and} \quad \phi^\infty_{AB} \le \phi_A^\infty + \phi_B^{R\infty}.$$

This result was recently extended in [11] to the case of non-commuting operators.

3. Parametric second-order symbols

In this section we consider the case where h is a second-order polynomial, that is, we consider the parametric problem

$$\mu u + \lambda e^{2sx}u - (\partial_x + \beta)^2 u = f, \quad x \in \mathbb{R}, \tag{3.1}$$

where $\lambda \in \Sigma_\pi$, $\mu \in \mathbb{C}$, $\beta \in \mathbb{R}$, $s \in \mathbb{R} \setminus \{0\}$ are fixed parameters. In this case we have

$$h(i\xi) = -(i\xi + \beta)^2 = \xi^2 - \beta^2 - 2i\beta\xi, \quad \xi \in \mathbb{R}.$$

The spectrum $\sigma(h(\partial_x))$ of the operator $h(\partial_x)$ in $L_p(\mathbb{R})$ is given by

$$\sigma(h(\partial_x)) = P_\beta := h(i\mathbb{R}) = \{\xi^2 - \beta^2 - 2i\beta\xi : \xi \in \mathbb{R}\},$$

a parabola with vertex in $-\beta^2$ opening to the right symmetric with respect to the real axis. We are going to derive an explicit characterization of the spectrum of $\lambda e^{2sx} + h(\partial_x)$ as well as an integral representation of the solutions of (3.1). Note that the change of variables $y = -x$ transforms (3.1) into itself, when replacing s by $-s$ and β by $-\beta$. Therefore we assume $s > 0$ in the sequel.

We introduce the variable transformation

$$u(x) = e^{-\beta x} w(\sqrt{\lambda} e^{sx}/s), \quad x \in \mathbb{R}.$$

Setting $z = \sqrt{\lambda} e^{sx}/s$ and $\nu^2 = \mu/s^2$, problem (3.1) transforms to

$$(z^2 + \nu^2) w(z) - z^2 w''(z) - z w'(z) = g(z), \quad \operatorname{Re} z > 0, \tag{3.2}$$

where

$$g(z) = z^{\beta/s} s^{\beta/s-2} \lambda^{-\beta/2s} f(s^{-1} \ln(sz/\sqrt{\lambda})).$$

(3.2) is nothing but the modified Bessel equation with parameter ν. A fundamental system of this equation is given by the modified Bessel function $I_\nu(z)$ and the McDonald function $K_\nu(z)$. The general solution of (3.2) is given by

$$w(z) = aI_\nu(z) + bK_\nu(z) + K_\nu(z) \int_0^z I_\nu(\sigma) g(\sigma) d\sigma/\sigma + I_\nu(z) \int_z^\infty K_\nu(\sigma) g(\sigma) d\sigma/\sigma, \tag{3.3}$$

where so far a, b are arbitrary numbers.

We are going to compute the spectrum of $A_{\lambda,\beta} = \lambda e^{2sx} - (\partial_x + \beta)^2$. Let us first look for eigenfunctions of $A_{\lambda,\beta}$, i.e., for the point spectrum $\sigma_p(A_{\lambda,\beta})$. So we have from (3.3) that $-\mu \in \sigma_p(A_{\lambda,\beta})$ if and only if a function of the form

$$u(x) = e^{-\beta x}[aI_\nu(\sqrt{\lambda} e^{sx}/s) + bK_\nu(\sqrt{\lambda} e^{sx}/s)], \quad x \in \mathbb{R},$$

belongs to $L_p(\mathbb{R})$ and $e^{2sx} u$ does so as well, since then we also have $u \in H_p^2(\mathbb{R})$ since $\omega + h(\partial_x) \in \mathcal{B}(H_p^2(\mathbb{R}), L_p(\mathbb{R}))$ is an isomorphism for $\omega > \beta^2$. In the sequel we are employing the asymptotics for $I_\nu(z)$ and $K_\nu(z)$ near zero and infinity which, for example, may be found in Abramowitz and Stegun [1]. For $Re\,\nu > 0$ these read

$$I_\nu(z) \sim e^z/\sqrt{2\pi z}, \quad K_\nu(z) \sim e^{-z}\sqrt{\pi/2z}, \quad \operatorname{Re} z > 0,\ |z| \to \infty, \tag{3.4}$$

and

$$I_\nu(z) \sim 2^{-\nu} z^\nu/\Gamma(\nu+1), \quad K_\nu(z) \sim 2^{\nu-1}\Gamma(\nu) z^{-\nu}, \quad \operatorname{Re} z > 0,\ z \to 0. \tag{3.5}$$

The estimates are uniform for $|\arg(z)| \le \theta < \pi/2$. From these asymptotics we may conclude $a = 0$, and set $b = 1$. Since $K_\nu(z)$ is exponentially decaying at infinity,

$$\int_0^\infty e^{p\gamma x} |u(x)|^p dx = c \int_1^\infty t^{(\gamma-\beta)p/s} |K_\nu(e^{i\varphi} t)|^p dt/t < \infty,$$

for each $p \in (1, \infty)$, $\beta, \gamma \in \mathbb{R}$ and $s > 0$ since $|\varphi| = |\arg \lambda/2| < \pi/2$. On the other hand,

$$\int_{-\infty}^0 e^{p\gamma x} |u(x)|^p dx = c \int_0^1 t^{(\gamma-\beta)p/s} |K_\nu(e^{i\varphi} t)|^p dt/t$$

is finite if and only if

$$\operatorname{Re} \nu p + (\beta - \gamma)p/s < 0, \quad \text{that is, iff} \quad \operatorname{Re} \sqrt{\mu} < \gamma - \beta.$$

Choosing $\gamma = 0$ and $\gamma = 2s$ we see that the function

$$u_\mu(x) = e^{-\beta x} K_\nu(\sqrt{\lambda} e^{sx}/s), \quad x \in \mathbb{R},$$

is an eigenfunction corresponding to the eigenvalue $-\mu$ of $A_{\lambda,\beta}$ if and only if $\operatorname{Re} \sqrt{\mu} < -\beta$. Note that this property is independent of $\lambda \in \Sigma_\pi$, $s > 0$ and $p \in (1, \infty)$.

Next observe that the dual operator $A^*_{\lambda,\beta}$ of $A_{\lambda,\beta}$ in $L_p(\mathbb{R})^* = L_{p'}(\mathbb{R})$ is precisely $A^*_{\lambda,\beta} = A_{\lambda,-\beta}$. Therefore we see from the above argument that $-\mu$ is an eigenvalue of $A^*_{\lambda,\beta}$ if and only if $\operatorname{Re} \sqrt{\mu} < \beta$. A dual eigenfunction is

$$u^*_\mu(x) = e^{\beta x} K_\nu(\sqrt{\lambda} e^{sx}/s), \quad x \in \mathbb{R}.$$

Thus the set $\operatorname{Re} \sqrt{-\mu} \leq |\beta|$ belongs to the spectrum of $A_{\lambda,\beta}$ for all $s > 0$, $1 < p < \infty$, and $\lambda \in \Sigma_\pi$, as long as $\beta \neq 0$. For $\beta = 0$ the point spectra of $A_{\lambda,\beta}$ and of $A^*_{\lambda,\beta}$ are both empty. We still claim that the set $\operatorname{Re} \sqrt{-\mu} = 0$, i.e., the set $\mu \in \mathbb{R}_+$, belongs to the spectrum of $A_{\lambda,\beta}$. In fact these values of $-\mu$ are approximate eigenvalues. Suppose on the contrary that $\mu + A_{\lambda,0}$ is invertible. Then $\mu + A_{\lambda,\beta}$ must be invertible for small $|\beta|$ as well, since $A_{\lambda,\beta}$ is a small perturbation of $A_{\lambda,0}$.

It is easy to identify the set $\operatorname{Re} \sqrt{-\mu} \leq |\beta|$. It is the filled up parabola $\mathbb{P}_\beta$, defined by

$$\mathbb{P}_\beta = \{\xi^2 - \beta^2 + \sigma - 2i\beta\xi : \ \xi \in \mathbb{R}, \ \sigma \geq 0\}.$$

The main result of this section is the following theorem which states that $\sigma(A_{\lambda,\beta}) = \mathbb{P}_\beta$, for all $p \in (1, \infty)$, $\lambda \in \Sigma_\pi$, $s > 0$, and $\beta \in \mathbb{R}$.

This is truly a remarkable result since the spectrum of $\lambda - (\partial_x + \beta)^2$ is the shifted parabola $\lambda + P_\beta$ instead!

Theorem 3.1. *Let $\lambda \in \Sigma_\pi$, $\beta \in \mathbb{R}$, $s > 0$, $1 < p < \infty$, and let $A_{\lambda,\beta}$ in $L_p(\mathbb{R})$ be defined according to*

$$A_{\lambda,\beta} = \lambda e^{2sx} - (\partial_x + \beta)^2,$$

with domain $\mathsf{D}(A_{\lambda,\beta}) = H^2_p(\mathbb{R}) \cap L_p(\mathbb{R}; e^{2sxp}dx)$. Then we have

$$\sigma(A_{\lambda,\beta}) = \mathbb{P}_\beta = \{\xi^2 - \beta^2 + \sigma - 2i\beta\xi : \ \xi \in \mathbb{R}, \ \sigma \geq 0\}.$$

If $\beta < 0$ then $\sigma_p(A_{\lambda,\beta}) = \mathring{\mathbb{P}}_\beta$ and $\sigma_r(A_{\lambda,\beta}) = \emptyset$, while for $\beta > 0$ we have $\sigma_r(A_{\lambda,\beta}) = \mathring{\mathbb{P}}_\beta$ and $\sigma_p(A_{\lambda,\beta}) = \emptyset$. The resolvent of $A_{\lambda,\beta}$ has the integral representation

$$(\mu + A_{\lambda,\beta})^{-1} f(x) = \int_{\mathbb{R}} k_{\lambda,\beta,\mu}(x,y) f(y)\,dy, \quad x \in \mathbb{R}, \ f \in L_p(\mathbb{R}),$$

where the kernel $k_{\lambda,\beta,\mu}$ is given by

$$k_{\lambda,\beta,\mu}(x,y) = s^{-1} e^{-\beta(x-y)} \begin{cases} K_{\sqrt{\mu}/s}(\sqrt{\lambda} e^{sx}/s) I_{\sqrt{\mu}/s}(\sqrt{\lambda} e^{sy}/s) & \text{for} \quad x > y \\ I_{\sqrt{\mu}/s}(\sqrt{\lambda} e^{sx}/s) K_{\sqrt{\mu}/s}(\sqrt{\lambda} e^{sy}/s) & \text{for} \quad x < y. \end{cases}$$

Finally, for every fixed $\phi \in (0,\pi)$ and $\mu \notin -\mathbb{P}_\beta$ there exists a positive constant M such that

$$|(\mu + A_{\lambda,\beta})^{-1}|_{\mathcal{B}(L_p)} + |\lambda e^{2sx}(\mu + A_{\lambda,\beta})^{-1}|_{\mathcal{B}(L_p)} \le M, \qquad \lambda \in \Sigma_\phi.$$

Proof. We first derive estimates for the kernel $k_{\lambda,\beta,\mu}$ for fixed $\mu \notin -\mathbb{P}_\beta$. From the asymptotics of I_ν and K_ν given in (3.4) and (3.5) we obtain for each $\theta < \pi/2$ a constant $c_{\nu,\theta} > 0$ such that

$$|I_\nu(z)| \le c_{\nu,\theta} \frac{|z|^{\operatorname{Re}\nu} e^{\operatorname{Re} z}}{(1+|z|)^{\operatorname{Re}\nu + 1/2}}, \quad |\arg(z)| \le \theta,$$

and

$$|K_\nu(z)| \le c_{\nu,\theta} \frac{|z|^{-\operatorname{Re}\nu} e^{-\operatorname{Re} z}}{(1+|z|)^{-\operatorname{Re}\nu + 1/2}}, \quad |\arg(z)| \le \theta.$$

With $\nu^2 = \mu/s^2$ these estimates yield for $x > y$

$$|k_{\lambda,\beta,\mu}(x,y)| \le C \frac{e^{-(\beta + s\operatorname{Re}\nu)(x-y)}}{1+|\sqrt{\lambda}|e^{sx}/s} \cdot \Big(\frac{1+|\sqrt{\lambda}|e^{sx}/s}{1+|\sqrt{\lambda}|e^{sy}/s}\Big)^{\operatorname{Re}\nu + 1/2} e^{-s^{-1}\operatorname{Re}\sqrt{\lambda}(e^{sx}-e^{sy})}.$$

The elementary inequality

$$\Big(\frac{1+a}{1+b}\Big)^\gamma \le c e^{\varepsilon(a-b)},$$

with $c > 0$ depending on $\varepsilon > 0$ and $\gamma > 0$ but not on $a > b > 0$, and the relation $\operatorname{Re}\sqrt{\lambda} \ge c'|\sqrt{\lambda}|$, valid for $|\arg \lambda| \le \phi < \pi$, then imply

$$|k_{\lambda,\beta,\mu}(x,y)| \le C \frac{e^{-(\beta + s\operatorname{Re}\nu)(x-y)}}{1+|\sqrt{\lambda}|e^{sx}} \cdot e^{-c_1|\sqrt{\lambda}|(e^{sx}-e^{sy})}, \quad x > y,$$

with some constants $C = C(\nu,\phi) > 0$ and $c_1 = c_1(\phi) > 0$. In a similar way we obtain for $x < y$

$$|k_{\lambda,\beta,\mu}(x,y)| \le C \frac{e^{-(-\beta + s\operatorname{Re}\nu)(y-x)}}{1+|\sqrt{\lambda}|e^{sy}} \cdot e^{-c_1|\sqrt{\lambda}|(e^{sy}-e^{sx})}, \quad x < y.$$

Combining these estimates leads to

$$|k_{\lambda,\beta,\mu}(x,y)| \le C \frac{e^{-(s\operatorname{Re}\nu - |\beta|)|x-y|}}{1+|\sqrt{\lambda}|e^{s\max\{x,y\}}} \cdot e^{-c_1|\sqrt{\lambda}||e^{sx}-e^{sy}|}, \quad x,y \in \mathbb{R}.$$

This shows $|k_{\lambda,\beta,\mu}(x,y)| \le C\kappa(x-y)$ with $\kappa(x) = e^{-(s\operatorname{Re}\nu - |\beta|)|x|}$, that is, the kernel $k_{\lambda,\beta,\mu}$ is dominated by a convolution kernel. Convolution with κ is L_p-bounded if $\kappa \in L_1(\mathbb{R})$, which is precisely the condition $\operatorname{Re}\nu s > |\beta|$, which in turn is equivalent to $\mu \notin -\mathbb{P}_\beta$. This shows that the resolvent of $A_{\lambda,\beta}$ is well-defined and L_p-bounded for all $\mu \notin \mathbb{P}_\beta$.

It remains to estimate $\lambda e^{2sx} k_{\lambda,\beta,\mu}$ to conclude $\sigma(A_{\lambda,\beta}) = \mathbb{P}_\beta$. Let

$$\gamma := \operatorname{Re}\nu s - |\beta| > 0 \quad \text{and} \quad \alpha := c_1|\sqrt{\lambda}| > 0.$$

Then we have

$$|\lambda e^{2sx} k_{\lambda,\beta,\mu}(x,y)| \le C e^{-\gamma|x-y|}[1 + \alpha e^{s\min\{x,y\}} e^{-\alpha|e^{sx}-e^{sy}|}].$$

This is clear in case $x < y$ and for $x > y$ it follows from

$$\alpha e^{sx} e^{-\alpha(e^{sx}-e^{sy})} = \alpha e^{sy} e^{-\alpha(e^{sx}-e^{sy})} + \alpha(e^{sx} - e^{sy}) e^{-\alpha(e^{sx}-e^{sy})}$$
$$\leq \alpha e^{sy} e^{-\alpha(e^{sx}-e^{sy})} + 1.$$

Thus we obtain

$$|\lambda e^{2sx} k_{\lambda,\beta,\mu}(x,y)| \leq Ce^{-\gamma|x-y|} + C\alpha e^{s\min\{x,y\}} e^{-\alpha|e^{sx}-e^{sy}|}, \quad x, y \in \mathbb{R}.$$

The first kernel on the right yields convolution with an L_1-function which is L_p-bounded, as before. The second one is symmetric, and we have

$$\int_{\mathbb{R}} s\alpha e^{s\min\{x,y\}} e^{-\alpha|e^{sx}-e^{sy}|} dy$$
$$\leq e^{-\alpha e^{sx}} \int_{-\infty}^{x} s\alpha e^{sy} e^{\alpha e^{sy}} dy + e^{\alpha e^{sx}} \int_{x}^{\infty} s\alpha e^{sy} e^{-\alpha e^{sy}} dy \leq 2.$$

This shows that the second kernel defines an integral operator which is simultaneously bounded in $L_1(\mathbb{R})$ and in $L_\infty(\mathbb{R})$, with bound independent of α. By the Riesz-Thorin interpolation theorem it is also bounded in $L_p(\mathbb{R})$, uniformly in $\alpha > 0$, i.e., in $\lambda \in \Sigma_\phi$ with $\phi < \pi$.

These arguments show that

$$(\mu + A_{\lambda,\beta})^{-1} L_p(\mathbb{R}) \subset L_p(\mathbb{R}; (1 + e^{2sxp})dx).$$

Since we know that the spectrum of $-(\partial_x + \beta)^2$ is the parabola P_β, we may conclude that $(\mu + A_{\lambda,\beta})^{-1} L_p(\mathbb{R}) \subset H^2_p(\mathbb{R})$, again with bound uniform in $\lambda \in \Sigma_\phi$. This concludes the proof of the theorem. $\square$

One should observe that our estimates are not uniform in μ, but they are of course so when μ is bounded. This then implies that near a point $\mu_0 \in P_\beta$ we have an estimate of the form

$$|(\mu - A_{\lambda,\beta})^{-1}|_{\mathcal{B}(L_p(\mathbb{R}))} \leq C/\mathrm{dist}(\mu, P_\beta), \quad \mu \notin \mathbb{P}_\beta,$$

which is optimal besides the constant C. However, we do not know whether a resolvent estimate of the form

$$|\mu(\mu - A_{\lambda,\beta})^{-1}|_{\mathcal{B}(L_p(\mathbb{R})} \leq C_\rho, \quad \mathrm{dist}(\mu, \mathbb{P}_\beta) \geq \rho,$$

for $\rho > 0$, uniformly in $\lambda \in \Sigma_\phi$, is valid. We are able to show this only for $\lambda > 0$.

4. Operator-valued symbols of second order

Having established uniform bounds for the operator families

$$\mathcal{K}_0 = \{(\mu + A_{\lambda,\beta})^{-1} : \lambda \in \Sigma_\phi\}, \quad \mathcal{K}_1 = \{\lambda e^{2sx}(\mu + A_{\lambda,\beta})^{-1} : \lambda \in \Sigma_\phi\} \subset \mathcal{B}(L_p(\mathbb{R}))$$

for every fixed $\phi \in (0, \pi)$, let us consider $\mathcal{R}$-boundedness of $\mathcal{K}_0$ and $\mathcal{K}_1$. Suppose we have established this property. Then holomorphy in λ permits to employ Theorem 2.1. We may replace λ by any sectorial operator D with bounded $\mathcal{H}^\infty$-calculus and angle $\phi_D^\infty < \pi$ which commutes with e^{2sx} and ∂_x. In particular, we are allowed to plug in $D = \partial_t$, or $D = \partial_t^\alpha$ for $\alpha \in (0,2)$ or, more generally, an anomalous diffusion operator $D = \partial_t^\alpha - \Delta_y$. The result then is that for all these different operators the spectrum of $L = De^{2sx} - (\partial_x + \beta)^2$ is contained in the parabola $\mathbb{P}_\beta$.

To prove $\mathcal{R}$-boundedness of $\mathcal{K}_0$ and $\mathcal{K}_1$ we use the domination theorem for kernel operators in $L_p(\mathbb{R})$ which says the following: if each of the kernels $k_{\lambda,\beta,\mu}$ for $\lambda \in \Sigma_\phi$, $\beta \in \mathbb{R}$ and $\mu \notin -\mathbb{P}_\beta$ fixed is pointwise bounded by a kernel κ_α and the set of kernel operators K_α, for α belonging to some index set, is $\mathcal{R}$-bounded then the family $\mathcal{K} = \{K_\alpha\}$ is so as well, see [6] for instance. Recall also that $\mathcal{R}$-boundedness is additive.

In our situation we have bounds for $\mathcal{K}_1$ in terms of a single convolution kernel $e^{-\gamma|x|}$, for some $\gamma > 0$, plus the family $\mathcal{L} := \{\alpha s e^{s\min\{x,y\}} e^{-\alpha|e^{sx}-e^{sy}|} : \ \alpha > 0\}$. Therefore, it is enough to show $\mathcal{R}$-boundedness in $L_p(\mathbb{R})$ of the family of kernel operators $\mathcal{L}$. To prove this the following lemma will be useful. We first introduce some more notation. Given any function $f \in L_{1,loc}(\mathbb{R})$, the maximal function Mf of f is defined by

$$(Mf)(x) := \sup_{a>0} \frac{1}{2a} \int_{x-a}^{x+a} |f(s)|\, ds, \quad x \in \mathbb{R}.$$

The result now reads as

Lemma 4.1.

(a) *Suppose $\phi \in C^2(\mathbb{R})$ is positive, nondecreasing and convex with $x\phi'(x) \to 0$ as $x \to -\infty$. Then the integral operator T_ϕ defined on $\mathcal{D}(\mathbb{R})$ by*

$$T_\phi f(x) = \phi(x)^{-1} \int_{-\infty}^{x} \phi'(y) f(y) dy, \quad x \in \mathbb{R},$$

satisfies $|T_\phi f(x)| \le 2(Mf)(x)$ for all $x \in \mathbb{R}$.

(b) *Suppose $\psi \in C^2(\mathbb{R})$ is positive, nonincreasing and $\ln\psi$ is concave with $x\psi(x) \to 0$ as $x \to \infty$. Then the integral operator S_ψ defined on $\mathcal{D}(\mathbb{R})$ by*

$$S_\psi f(x) = \frac{-\psi'(x)}{\psi(x)^2} \int_x^\infty \psi(y) f(y) dy, \quad x \in \mathbb{R},$$

satisfies $|S_\psi f(x)| \le 2(Mf)(x)$ for all $x \in \mathbb{R}$.

Proof. (a) Let $f \in \mathcal{D}(\mathbb{R})$ be given. Then we have

$$\begin{aligned} |T_\phi f(x)| &\le \phi(x)^{-1} \int_{-\infty}^{x} \phi'(y)|f(y)|dy = \phi(x)^{-1} \int_{-\infty}^{x} \phi''(y) \int_y^x |f(s)|dsdy \\ &\le 2(Mf)(x)\phi(x)^{-1} \int_{-\infty}^{x} \phi''(y)(x-y)dy \\ &= 2(Mf)(x)\phi(x)^{-1} \int_{-\infty}^{x} \phi'(y)dy \le 2(Mf)(x). \end{aligned}$$

These estimates can be justified by first integrating over a finite interval and then letting the lower limit go to $-\infty$. It should be noted that $\int_{-\infty}^x \phi'(y)\, dy \le \phi(x)$ since ϕ is positive and nondecreasing.

(b) Similarly we obtain

$$
\begin{aligned}
|S_\psi f(x)| &\le \frac{-\psi'(x)}{\psi(x)^2}\int_x^\infty \psi(y)|f(y)|dy = \frac{\psi'(x)}{\psi(x)^2}\int_x^\infty \psi'(y)\int_x^y |f(s)|dsdy\\
&\le 2(Mf)(x)\frac{\psi'(x)}{\psi(x)^2}\int_x^\infty \psi'(y)(y-x)dy\\
&= 2(Mf)(x)\frac{-\psi'(x)}{\psi(x)^2}\int_x^\infty \psi(y)dy\\
&\le 2(Mf)(x)\psi(x)^{-1}\int_x^\infty -\psi'(y)dy = 2(Mf)(x),
\end{aligned}
$$

where in the last inequality we used that $-\psi'(x)/\psi(x) = -(d/dx)\ln\psi(x)$ is increasing, since $\ln\psi(x)$ is concave by assumption. □

Now we can prove the following result.

Proposition 4.2. *Let $\beta \in \mathbb{R}$, $s > 0$, $1 < p < \infty$, and suppose $\mu \notin -\mathbb{P}_\beta$. Then the operator families*

$$\mathcal{K}_0 = \{(\mu + A_{\lambda,\beta})^{-1} :\ \lambda \in \Sigma_\phi\} \subset \mathcal{B}(L_p(\mathbb{R}))$$

and

$$\mathcal{K}_1 = \{\lambda e^{2sx}(\mu + A_{\lambda,\beta})^{-1} :\ \lambda \in \Sigma_\phi\} \subset \mathcal{B}(L_p(\mathbb{R}))$$

are $\mathcal{R}$-bounded for every fixed $\phi \in (0,\pi)$.

Proof. The proof uses the characterization of $\mathcal{R}$-boundedness of integral operators in L_p-spaces by means of square function estimates and proceeds like the proof of Theorem 4.8 in Denk, Hieber and Prüss [6]; see also Clément and Prüss [3] where this argument appears for the first time.

Take any functions $f_i \in L_p(\mathbb{R})$ and any numbers $\lambda_i \in \Sigma_\phi$ and let K_i denote the integral operator with kernel $\lambda_i e^{2sx}k_{\lambda_i,\beta,\mu}(x,y)$. Then with $\phi_i(x) = e^{\alpha_i e^{sx}}$, $\psi_i(x) = e^{-\alpha_i e^{sx}}$, $\alpha_i = c_1|\sqrt{\lambda_i}| > 0$, Lemma 4.1 yields

$$|K_i f_i(x)| \le C(Mf_i)(x), \qquad \text{for a.a. } x \in \mathbb{R},$$

hence

$$\sum_i |K_i f_i(x)|^2 \le C\sum_i |Mf_i(x)|^2, \qquad \text{for a.a. } x \in \mathbb{R},$$

where $C > 0$ denotes a constant that is independent of f_i and λ_i. Since the maximal operator M is bounded in $L_p(\mathbb{R};l_2)$ for $1 < p < \infty$, see [19, Theorem II.1.1], we may continue

$$|(\sum_i |K_i f_i|^2)^{1/2}|_{L_p(\mathbb{R})} \le C|(\sum_i |Mf_i|^2)^{1/2}|_{L_p(\mathbb{R})} \le C|(\sum_i |f_i|^2)^{1/2}|_{L_p(\mathbb{R})}.$$

This implies that the family $\mathcal{K}_1$ satisfies a square function estimate, and it follows from [6, Remark 3.2(4)] that $\mathcal{K}_1$ is $\mathcal{R}$-bounded. $\mathcal{K}_0$ is $\mathcal{R}$-bounded since it is dominated by a single L_p-bounded kernel operator, see [6, Proposition 3.5] for instance. □

We can now state the following result, alluded to in the introduction to this section.

Theorem 4.3. *Let $\beta \in \mathbb{R}$, $s > 0$, $1<p<\infty$, and let D denote an invertible sectorial operator on a Banach space X with bounded $\mathcal{H}^\infty$-calculus and angle $\phi_D^\infty < \pi$. Let L on $L_p(\mathbb{R}; X)$ be defined by means of*

$$Lu = De^{2sx} - (\partial_x + \beta)^2,$$

with natural domain

$$\mathsf{D}(L) = \{u \in H_p^2(\mathbb{R}; X) : \ e^{2sx} u \in L_p(\mathbb{R}; \mathsf{D}(D))\}.$$

Then $\sigma(L) \subset \mathbb{P}_\beta$, and for any $\omega > \beta^2$, $\omega + L$ is invertible, sectorial, and admits a bounded $\mathcal{H}^\infty$-calculus in $L_p(\mathbb{R}; X)$.

Proof. We may conclude from Theorem 3.1 and Proposition 4.2 that $\sigma(L) \subset \mathbb{P}_\beta$, see the arguments given at the beginning of the current section. Consequently

$$\mu + L \quad \text{is invertible for every } \mu \notin -\mathbb{P}_\beta. \tag{4.1}$$

We now consider the operator A given by

$$A := De^{2sx}, \quad \mathsf{D}(A) := \{u \in L_p(\mathbb{R}, X) : e^{2sx} u \in L_p(\mathbb{R}, \mathsf{D}(D)\}.$$

A is the product of D with the multiplication operator $M := e^{2sx}$, which has an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus with $\phi_M^{R\infty} = 0$ on $L_p(\mathbb{R})$, and hence also on $L_p(\mathbb{R}, X)$. It follows from the remarks at the end of Section 2 that $A \in \mathcal{H}^\infty(\mathbb{R}, X)$ with $\phi_A^\infty \le \phi_D^\infty$. Next we consider the operator B, defined by

$$B := \delta_0 - (\partial_x - \beta)^2, \quad \mathsf{D}(B) = H_p^2(\mathbb{R}),$$

where $\delta_0 > \beta^2$. It follows from Mikhlin's theorem that B is sectorial and admits an $\mathcal{H}^\infty$-calculus on $L_p(\mathbb{R})$. Moreover, we can also conclude that $\delta_0 > \beta^2$ can be chosen in such a way that $\phi_D^\infty + \phi_B^\infty < \pi$. Since $L_p(\mathbb{R})$ has property α we obtain, in addition, that $\phi_D^\infty + \phi_B^{R\infty} < \pi$. This result also holds for the canonical extension of B to $L_p(\mathbb{R}, X)$. The same arguments as in [17, Section 5] now show that there exists a number $\omega_0 \ge 0$ such that

$$\omega_0 + A + B \quad \text{is sectorial and admits an } \mathcal{H}^\infty\text{-calculus on } L_p(\mathbb{R}, X). \tag{4.2}$$

The remaining assertions of Theorem 4.3 are now a consequence of (4.1)–(4.2) and [6, Proposition 2.7]. □

Remarks 4.4. (a) Suppose $D = \partial_t$, with domain $\mathsf{D}(\partial_t) = \{u \in W_p^1(J) : u(0) = 0\}$. Then it can be shown that, in fact, $\sigma(L) = \mathbb{P}_\beta$. This follows from the property that every number $-\mu \in \mathring{\mathbb{P}}_\beta$ is an eigenvalue of either $A_{\lambda,\beta}$ or its dual, with eigenfunction

$$u_\mu(\lambda, x) = e^{\pm\beta x} K_\nu(\sqrt{\lambda} e^{sx}/s), \quad \lambda \in \Sigma_\pi, \ x \in \mathbb{R},$$

see Section 3. Taking the inverse Laplace transform with respect to λ over an appropriate contour will provide an eigenfunction of L, or its dual, with eigenvalue $-\mu$.

(b) A correspondig result to Theorem 4.3 can also be stated for operators of the form $D + (\omega - (\partial_x + \beta)^2)e^{-2sx}$, but we leave this to the interested reader.

5. Parabolic equations with dynamic boundary conditions on wedges and angles

In this section we consider an application of our main results to the diffusion equation on a domain of wedge or angle type, that is, on the domain $G = \mathbb{R}^m \times C_\alpha$, where $m \in \mathbb{N}_0$, and for $\alpha \in (0, 2\pi)$, C_α denotes the angle

$$C_\alpha = \{x = (r\cos\phi, r\sin\phi) : \ r > 0, \ \phi \in (0, \alpha)\}.$$

The boundary $\Gamma = \partial G$ then consists of two faces

$$\Gamma_0 = \{(y, r, 0); \ y \in \mathbb{R}^m, \ r > 0\}, \quad \Gamma_\alpha = \{(y, r\cos\alpha, r\sin\alpha) : \ y \in \mathbb{R}^m, \ r > 0\}.$$

We consider the problem

$$\left\{\begin{aligned} \partial_t u - \Delta u &= f_1 && \text{in } G \times (0,T) \\ u &= f_2 && \text{on } \Gamma_\alpha \times (0,T) \\ \partial_t u + \partial_\nu u &= f_3 && \text{on } \Gamma_0 \times (0,T) \\ u|_{t=0} &= u_0 && \text{on } G. \end{aligned}\right. \tag{5.1}$$

Here $m \in \mathbb{N}_0$ and ν denotes the outer normal for G at Γ. The function f_1 is given in a weighted L_p-space, i.e.,

$$f_1 \in L_p(J \times \mathbb{R}^m; L_p(C_\alpha; |x|^\gamma dx)),$$

where $\gamma \in \mathbb{R}$ will be chosen appropriately, and $J = (0,T)$. The functions f_2 and f_3 are supposed to belong to certain trace spaces.
It is natural to introduce polar coordinates in the x-variables, $x = (r\cos\phi, r\sin\phi)$ where $\phi \in (0, \alpha)$ and $r > 0$. Then the diffusion operator $\partial_t - \Delta$ transforms into

$$\partial_t - \Delta_y - [\partial_r^2 + \frac{1}{r}\partial_r] - \frac{\partial_\phi^2}{r^2},$$

where y denotes the variable in $\mathbb{R}^m$, and Δ_y is the Laplacian in the y-variables. The underlying space for the function f_1 now is

$$f_1 \in L_p(J \times \mathbb{R}^m \times (0,\alpha); L_p(\mathbb{R}_+; r^{\gamma+1} dr)).$$

It is also natural to employ the Euler transformation $r = e^x$ where now $x \in \mathbb{R}$. Setting

$$u(t, y, \phi, r) = r^\beta v(t, y, \phi, \ln r),$$

we arrive at the following problem for the unknown function v.

$$\left\{\begin{aligned} e^{2x}(\partial_t - \Delta_y)v + P(\partial_x)v - \partial_\phi^2 v &= g_1 && \text{in } (0,T) \times \mathbb{R}^m \times (0,\alpha) \times \mathbb{R} \\ v &= g_2 && \text{on } (0,T) \times \mathbb{R}^m \times \{\alpha\} \times \mathbb{R} \\ e^x \partial_t v - \partial_\phi v &= g_3 && \text{on } (0,T) \times \mathbb{R}^m \times \{0\} \times \mathbb{R} \\ v|_{t=0} &= v_0 && \text{on } \mathbb{R}^m \times (0,\alpha) \times \mathbb{R} \end{aligned}\right. \tag{5.2}$$

where $g_1(t,y,\phi,x) = e^{(2-\beta)x} f_1(t,y,\phi,e^x)$ and

$$g_2(t,y,x) = e^{-\beta x} f_2(t,y,e^x), \quad g_3(t,y,x) = e^{(1-\beta)x} f_3(t,y,e^x).$$

The differential operator $P(\partial_x)$ is given by the polynomial $P(z) = -(z+\beta)^2$, as a simple computation shows. The resulting equations are now defined in a smooth domain, but they contain the (non-standard) differential operators $e^{sx}\partial_t$, $s = 1, 2$, and $e^{2x}\Delta_y$. We observe that these operators do not commute with $P(\partial_x)$.

Next we note that

$$\int_{\mathbb{R}} |g_1(t,y,\phi,x)|^p dx = \int_0^\infty |r^{2-\beta} f_1(t,y,\phi,r)|^p dr/r < \infty,$$

in case we choose $p(2-\beta) = \gamma+2$, that is, $\beta = 2-(\gamma+2)/p$. Making this choice of β, we can remove the weight and work in the unweighted base space

$$X := L_p(J \times \mathbb{R}^m \times (0,\alpha) \times \mathbb{R}).$$

We want to extract the boundary symbol for problem (5.2). For this purpose we define an operator A_β in $X = L_p(J \times \mathbb{R}^m \times \mathbb{R})$, $J = (0,T)$, by means of

$$(A_\beta u)(t,y,x) = ((\partial_t - \Delta_y)e^{2x} - (\partial_x + \beta)^2)u, \quad (t,y,x) \in J \times \mathbb{R}^m \times \mathbb{R},$$

with domain

$$\begin{aligned}\mathsf{D}(A_\beta) \;=\; & L_p(J \times \mathbb{R}^m; H^2_p(\mathbb{R})) \cap {}_0H^1_p(J; L_p(\mathbb{R}^m; L_p(\mathbb{R}; e^{2xp}dx))) \\ & \cap H^2_p(\mathbb{R}^m; L_p(J; L_p(\mathbb{R}; e^{2xp}dx))).\end{aligned}$$

Then the solution of the homogeneous problem

$$\left\{\begin{aligned} e^{2x}(\partial_t - \Delta_y)v + P(\partial_x)v - \partial_\phi^2 v = 0 &\quad \text{in } (0,T) \times \mathbb{R}^m \times (0,\alpha) \times \mathbb{R} \\ v = 0 &\quad \text{on } (0,T) \times \mathbb{R}^m \times \{\alpha\} \times \mathbb{R} \\ v = \rho &\quad \text{on } (0,T) \times \mathbb{R}^m \times \{0\} \times \mathbb{R} \\ v|_{t=0} = 0 &\quad \text{on } \mathbb{R}^m \times (0,\alpha) \times \mathbb{R} \end{aligned}\right.$$

with Dirichlet datum $v = \rho$ on $J \times \mathbb{R}^m \times \{0\} \times \mathbb{R}$ is given by

$$v(\phi) = \varphi(A_\beta,\phi)\rho, \quad \varphi(z,\phi) = \sinh((\alpha-\phi)\sqrt{z})/\sinh(\alpha\sqrt{z}), \quad \phi \in (0,\alpha).$$

Evaluating the normal derivative and inserting into the dynamic boundary condition yields

$$e^x \partial_t \rho + \psi(A_\beta)\rho = g, \quad \rho|_{t=0} = 0, \quad (t,y,x) \in J \times \mathbb{R}^m \times \mathbb{R} \tag{5.3}$$

where $\psi(z) = \sqrt{z}\coth(\alpha\sqrt{z})$.

We want to study the boundary symbol $e^x\partial_t + \psi(A_\beta)$ in the base space $X := L_p(J \times \mathbb{R}^m \times \mathbb{R})$. For this purpose we note that according to Theorem 4.3 the operator $A_\beta + \delta$ admits a bounded $\mathcal{H}^\infty$-calculus for each $\delta > \beta^2$. Moreover, it is not difficult to see that $A_\beta + \delta$ is m-accretive on X whenever $\delta > \beta^2$. The function ψ is meromorphic on $\mathbb{C}$ with poles in

$$\{z_k = -r_k^2 := -k^2(\pi/\alpha)^2 : k \in \mathbb{N}\}$$

and $\psi(z) \sim \sqrt{z}$ as $z \to \infty$, provided $|\arg z| \leq \theta < \pi$. It follows that $\psi(A_\beta)$ is a well-defined, closed linear operator with $\mathsf{D}(\psi(A_\beta)) = \mathsf{D}((A_\beta + \delta)^{1/2})$, see [6, Section 2.1] for more details.
The function ψ admits the following representation as a series

$$\psi(z) = \frac{1}{\alpha}\Big[1 + 2z \sum_{k=1}^{\infty} \frac{1}{z + r_k^2}\Big], \quad z \notin \{-r_j^2 : j \in \mathbb{N}\}.$$

Inserting A_β into this representation we obtain

$$\begin{aligned}\alpha\, \psi(A_\beta) &= 1 + 2\sum_{k=1}^{\infty} A_\beta (A_\beta + r_k^2)^{-1} \\ &= 1 + 2\sum_{k=1}^{\infty} (A_\beta + \beta^2)(A_\beta + r_k^2)^{-1} - 2\beta^2 \sum_{k=1}^{\infty} (A_\beta + r_k^2)^{-1}.\end{aligned}$$

Employing the semi-inner product $(\cdot\,,\,\cdot)$ in X we estimate as follows.

$$\begin{aligned}\alpha\big(\psi(A_\beta)u, u\big) &\geq |u|^2 + 2\sum_{k=1}^{\infty} \big((A_\beta + \beta^2)(A_\beta + \beta^2 + (r_k^2 - \beta^2))^{-1}u, u\big) \\ &\qquad - 2\beta^2 \sum_{k=1}^{\infty} |(A_\beta + \beta^2 + (r_k^2 - \beta^2))^{-1}u||u| \\ &\geq \Big[1 - 2\beta^2 \sum_{k=1}^{\infty} (r_k^2 - \beta^2)^{-1}\Big]|u|^2 \\ &\geq \Big[1 - \frac{\alpha|\beta|}{\pi - \alpha|\beta|}\Big]|u|^2 = \frac{\pi - 2\alpha|\beta|}{\pi - \alpha|\beta|}|u|^2,\end{aligned}$$

provided $\frac{1}{(k+1)\pi - \alpha|\beta|} \leq \frac{1}{k\pi + \alpha|\beta|}$ for all $k \in \mathbb{N}$, which is equivalent to $|\beta| \leq \pi/2\alpha$.
Thus if β is restricted to the range $|\beta| < \pi/2\alpha$ then $\psi(A_\beta)$ is strictly accretive in X.

We remark that the condition $|\beta| < \pi/2\alpha$ is also necessary for $\psi(A_\beta)$ to be strictly accretive. In order to see this, recall that $\psi(\sigma(A_\beta)) \subset \sigma(\psi(A_\beta))$ according to the (weak) spectral mapping theorem. Thus for $\psi(A_\beta)$ to be strictly accretive, $\psi(\sigma(A_\beta))$ must be contained in the set $[\mathrm{Re}\, z \geq c]$ with $c > 0$. Note that the curves $-(\eta + i\xi)^2$ with $0 \leq \eta \leq |\beta|$ fill up the parabola $\mathbb{P}_\beta$, the spectrum of A_β. The function

$$\mathrm{Re}\, \psi(-(\eta + i\xi)^2) = \frac{\xi \sinh(2\alpha\xi) + \eta \sin(2\alpha\eta)}{\cosh(2\alpha\xi) - \cos(2\alpha\eta)}$$

is strictly positive for all $\xi \in \mathbb{R}$ and $0 \leq |\eta| \leq |\beta|$ if and only if $\sin(2\alpha|\eta|) > 0$, the latter condition being equivalent to $|\beta| < \pi/2\alpha$. Thus $\psi(\mathbb{P}_\beta) \subset [\mathrm{Re} z > 0]$ if and only if $|\beta| < \pi/2\alpha$.

It should also be observed that in case we assume a Neumann condition on $\phi = \alpha$, $\psi(z) = \sqrt{z}\tanh(\alpha\sqrt{z})$. In this case

$$\operatorname{Re}\psi(-(\eta + i\xi)^2) = \frac{\xi\sinh(2\alpha\xi) - \eta\sin(2\alpha\eta)}{\cosh(2\alpha\xi) + \cos(2\alpha\eta)},$$

which does not have positive real part for $\xi \in \mathbb{R}$, for no values of η. Thus in this case, $\psi(A_\beta)$ fails to be accretive.

We will now show that there exists a sufficiently large positive number ω_0 such that $\omega_0 + \partial_t e^x + \psi(A_\beta)$, with domain $\mathsf{D}(\partial_t e^x) \cap \mathsf{D}(\psi(A_\beta))$, is invertible, sectorial and admits a bounded $\mathcal{H}^\infty$-calculus on X. Since $e^x\partial_t + \psi(A_\beta)$ is strictly accretive, we can conclude that $e^x\partial_t + \psi(A_\beta)$ is in fact invertible and m-accretive, and hence sectorial on X. It then follows from [6, Proposition 2.7] that $\partial_t e^x + \psi(A_\beta)$ admits a bounded $\mathcal{H}^\infty$-calculus on X as well.

To prove the remaining statement we use, once more, the result on sums of non-commuting operators, Theorem 2.2. For this purpose we have to estimate the relevant commutator in the Labbas-Terreni condition (2.3). We will need the following auxiliary result for the commutator of $(z - A_\beta)^{-1}$ with the multiplication operator e^x.

Lemma 5.1. *Suppose $z \in \rho(A_\beta) \cap \rho(A_{\beta-1})$. Then*

$$[(z - A_\beta)^{-1}, e^x]e^{-x} = \big((z - A_\beta)^{-1} - (z - A_{\beta-1})^{-1}\big) \text{ on } X.$$

Proof. An easy computation shows that $(z - A_{\beta-1})e^x v = e^x(z - A_\beta)v$ for every function $v \in \mathcal{D} := \mathcal{D}((0,T] \times \mathbb{R}^m \times \mathbb{R})$. Applying $(z - A_{\beta-1})^{-1}$ to this equation gives

$$e^x v = (z - A_{\beta-1})^{-1} e^x (z - A_\beta)v, \quad v \in \mathcal{D}.$$

Substituting $v = (z - A_\beta)^{-1}(z - A_\beta)v$ on the left side and then replacing $(z - A_\beta)v$ by $e^{-x}(z - A_\beta)v$ yields

$$e^x(z - A_\beta)^{-1}e^{-x}(z - A_\beta)v = (z - A_{\beta-1})^{-1}(z - A_\beta)v, \quad v \in \mathcal{D}.$$

The assertion now follows from the fact that $\{(z - A_\beta)v : v \in \mathcal{D}\}$ is dense in X. □

In order to check the commutator condition (2.3) we first remove some poles from ψ such that the remainder ψ_0 is holomorphic in a sector $-a + \Sigma_\phi$, where

$$-a < -b := \min\{-(\beta - 1)^2, -\beta^2\}, \quad \phi \in (0, \pi).$$

Observing that ψ has first-order poles at $-r_k^2$ with corresponding residues $-(2/\alpha)r_k^2$ we can take

$$\psi_0(z) := \psi(z) + \frac{2r_1^2}{\alpha} \cdot \frac{1}{z + r_1^2} + c\,,$$

with c a constant. The condition $|\beta| < \pi/\alpha$ ensures that ψ_0 is in fact holomorphic on an open set containing the closure of $-a + \Sigma_\phi$ for $-a \in (-r_2^2, -b)$ and any $\phi \in (0, \pi)$. By choosing c big enough we can, moreover, arrange that ψ_0 maps the sector $-a+\Sigma_\phi$, with $\phi<\pi$, into a sector Σ_θ with $\theta<\pi/2$. The difference $\psi(A_\beta)-\psi_0(A_\beta)$ is bounded, as one easily verifies. Then we fix $\eta>0$ and compute

$$\begin{aligned}&(\eta+\partial_t e^x)(\lambda+\eta+\partial_t e^x)^{-1}[(\eta+\partial_t e^x)^{-1},(\mu+\psi_0(A_\beta))^{-1}]\\&=(\lambda+\eta+\partial_t e^x)^{-1}[(\mu+\psi_0(A_\beta))^{-1},e^x]e^{-x}\cdot\partial_t e^x(\eta+\partial_t e^x)^{-1},\end{aligned}$$

which gives

$$\begin{aligned}&\left|(\eta+\partial_t e^x)(\lambda+\eta+\partial_t e^x)^{-1}[(\eta+\partial_t e^x)^{-1},(\mu+\psi_0(A_\beta))^{-1}]\right|_{\mathcal{B}(X)}\\&\le C_\eta\,(1+|\lambda|)^{-1}\cdot\left|[(\mu+\psi_0(A_\beta))^{-1},e^x]e^{-x}\right|_{\mathcal{B}(X)}.\end{aligned}$$

Next taking $\Gamma=-a+\partial\Sigma_\phi$, the boundary of the sector $-a+\Sigma_\phi$, with ϕ appropriate, and using Lemma 5.1 we get

$$\begin{aligned}&\left|[(\mu+\psi_0(A_\beta))^{-1},e^x]e^{-x}\right|_{\mathcal{B}(X)}=\left|\frac{1}{2\pi i\mu}\int_\Gamma\frac{\psi_0(z)}{\mu+\psi_0(z)}[(z-A_\beta)^{-1},e^x]e^{-x}\,dz\right|_{\mathcal{B}(X)}\\&\le\frac{C}{|\mu|}\int_\Gamma\left|\frac{\psi_0(z)}{\mu+\psi_0(z)}\right|\cdot\left|(z-A_\beta)^{-1}-(z-A_{\beta-1})^{-1}\right|_{\mathcal{B}(X)}|dz|.\end{aligned}$$

Since $A_\beta-A_{\beta-1}=-2\partial_x-2\beta+1$ we obtain

$$\begin{aligned}\left|(z-A_\beta)^{-1}-(z-A_{\beta-1})^{-1}\right|_{\mathcal{B}(X)}&\le\left|(z-A_\beta)^{-1}(A_\beta-A_{\beta-1})(z-A_{\beta-1})^{-1}\right|_{\mathcal{B}(X)}\\&\le C(1+|z|)^{-3/2}.\end{aligned}$$

This implies

$$\begin{aligned}&\left|[(\mu+\psi_0(A_\beta))^{-1},e^x]e^{-x}\right|_{\mathcal{B}(X)}\le\frac{C}{|\mu|}\int_\Gamma\left|\frac{\psi_0(z)}{\mu+\psi_0(z)}\right|\frac{|dz|}{(1+|z|)^{3/2}}\\&\le\frac{C}{|\mu|}\int_\Gamma\frac{|dz|}{(|\mu|+|z|^{1/2})(1+|z|)}\le\frac{C_\varepsilon}{|\mu|^{2-\varepsilon}}\,,\end{aligned}$$

since $\psi_0(z)\sim\sqrt{z}$. Thus the assumptions of Theorem 2.2 are satisfied for

$$A:=\eta+\partial_t e^x\quad\text{and}\quad B:=\psi_0(A_\beta),$$

with $\alpha=0$ and $\beta=1-\varepsilon$, for each $\varepsilon>0$. Indeed, observe that $\psi_0(A_\beta)$ is sectorial with angle strictly smaller than $\pi/2$. This follows from the fact that ψ_0 maps the sector $-a+\Sigma_\phi$ into a sector Σ_θ with $\theta<\pi/2$. Hence the parabolicity condition holds. Moreover, it is clear that $\psi_0(A_\beta)$ admits a bounded $\mathcal{H}^\infty$-calculus on X since $A_\beta+\delta$ does so according to Theorem 4.3. We may conclude that there is a number $\omega_0>0$ such that $\omega_0+\partial_t e^x+\psi_0(A_\beta)$ with natural domain is closed, invertible, sectorial, and admits an $\mathcal{H}^\infty$-calculus on X. Perturbing by the bounded linear operator $\psi(A_\beta)-\psi_0(A_\beta)$ we see that the same result holds for $\omega_0+\partial_t e^x+\psi(A_\beta)$, possibly after choosing a larger number ω_0.

We summarize our considerations in

Theorem 5.2. *Let $1<p<\infty$ and assume $|\beta|<\pi/2\alpha$. Then for each $g\in X:=L_p(J\times\mathbb{R}^m\times\mathbb{R})$, (5.3) admits a unique solution ρ in*

$$L_p(J\times\mathbb{R}^m;H^r_p(\mathbb{R}))\cap L_p(J;H^1_p(\mathbb{R}^m;L_p(\mathbb{R};e^{xp}dx)))\cap H^1_p(J;L_p(\mathbb{R}^m;L_p(\mathbb{R};e^{xp}dx))).$$

There is a constant $M > 0$, independent of g, such that

$$|e^{sx}\partial_t\rho|_X + |\rho|_{L_p(J\times\mathbb{R}^m;H_p^r(\mathbb{R}))} + |\rho|_{L_p(J;H_p^1(\mathbb{R}^m;L_p(\mathbb{R};e^{xp}dx)))} \leq M|g|_X.$$

The operator $L = \partial_t e^{sx} + \psi((\partial_t - \Delta_y)e^{2x} - (\partial_x + \beta)^2)$ admits a bounded $\mathcal{H}^\infty$-calculus on X.

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Maximal L_p-regularity and Long-time Behaviour of the Non-isothermal Cahn-Hilliard Equation with Dynamic Boundary Conditions

Jan Prüss and Mathias Wilke

Dedicated to Philippe Clément

Abstract. In this paper we investigate the nonlinear Cahn-Hilliard equation with nonconstant temperature and dynamic boundary conditions. We show maximal L_p-regularity for this problem with inhomogeneous boundary data. Furthermore we show global existence and use the Lojasiewicz-Simon inequality to show that each solution converges to a steady state as time tends to infinity, as soon as the potential Φ and the latent heat λ satisfy certain growth conditions.

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1. Introduction and the model

During the last years there has been much interest in so-called phase-field models arising from the theory of phase transitions. Basically there are four models, namely the Cahn-Allen equation, the Cahn-Hilliard equation, the Penrose-Fife system and the Caginalp system. There is a quite large literature concerning these phase-field models, especially for the Cahn-Hilliard equation

$$\partial_t \psi = \Delta \mu, \quad \mu = -\Delta \psi + \Phi'(\psi), \quad t > 0,\ x \in \Omega, \tag{1.1}$$

with constant temperature. Here ψ is the so-called order-parameter for the investigated physical system, e.g., the mass density or the magnetic flux, etc. The functions μ and Φ are the chemical and physical potentials, respectively.

Consider the free energy density

$$F(\psi,\vartheta)=\frac{1}{2}|\nabla\psi|^2+\Phi(\psi)-\lambda(\psi)\vartheta-\frac{1}{2}\vartheta^2,$$

where we assume that the relative temperature ϑ varies in time and space and Φ,λ are given functions. One obtains the chemical potential μ and the internal energy e by differentiating F with respect to ψ and ϑ, respectively. We then have

$$\mu=\frac{\partial F}{\partial\psi}(\psi,\vartheta)=-\Delta\psi+\Phi'(\psi)-\lambda'(\psi)\vartheta$$

and

$$e=-\frac{\partial F}{\partial\vartheta}(\psi,\vartheta)=\lambda(\psi)+\vartheta.$$

To obtain kinetic equations we assume that the order parameter ψ and the internal energy e are conserved quantities. The according conservation laws read as follows

$$\partial_t\psi+\operatorname{div}\mathcal{F}=0,\quad \partial_t e+\operatorname{div}q=0.$$

Here q is the heat flux, which in this paper is assumed to be given by Fourier's law $q=-\nabla\vartheta$ and $\mathcal{F}$ denotes the phase flux of the order parameter ψ which is assumed to be of the form $\mathcal{F}=-\nabla\mu$. Thus the kinetic equations read as follows

$$\partial_t\psi-\Delta\mu=0,\quad \mu=-\Delta\psi+\Phi'(\psi)-\lambda'(\psi)\vartheta,\quad t\in J,\ x\in\Omega,\tag{1.2}$$

$$\partial_t\vartheta+\lambda'(\psi)\partial_t\psi-\Delta\vartheta=0,\quad t\in J,\ x\in\Omega,\tag{1.3}$$

where $J=[0,T]$ or $J=\mathbb{R}_+$, and $\Omega\subset\mathbb{R}^n$ is a bounded domain with boundary $\Gamma=\partial\Omega$ of class C^4. To ensure that ψ is a conserved quantity we take Neumann boundary conditions for μ, i.e., $\partial_\nu\mu=0$. Concerning ϑ we will use Robin boundary conditions, namely $\alpha\vartheta+\partial_\nu\vartheta=0$, where $\alpha\geq 0$ is a constant. Since (1.2) is an equation of fourth order, we need another boundary condition for ψ. Usually one uses further classical boundary conditions, i.e., $\partial_\nu\psi=0$ etc. (cf. also [1]). Recently, to account for boundary effects, the authors in [7] proposed a dynamic boundary condition of the form

$$\partial_t\psi-\sigma_s\Delta_\Gamma\psi+\gamma\partial_\nu\psi+\kappa\psi=0,\tag{1.4}$$

where $\sigma_s,\gamma>0$, $\kappa\geq 0$. Well-posedness and maximal L_p-regularity of the classical Cahn-Hilliard equation (1.1) with dynamic boundary conditions have been shown in [8]. Concerning the asymptotic behavior of solutions of (1.1) subject to dynamic boundary conditions we refer to the references [2] and [11]. There the authors made use of the Lojasiewicz-Simon inequality to show that every solution converges to a steady state as time tends to infinity. So far there seems to be no result on well-posedness and asymptotic behavior of (1.2), (1.3) with dynamic boundary conditions. This is the subject of this paper. For results concerning well-posedness and asymptotic behavior of solutions of (1.2) and (1.3) with classical boundary conditions, we refer to [1], [5], [6], [13] and the references cited therein. Here the system we investigate is

$$
\begin{aligned}
\partial_t\psi - \Delta\mu = f_1, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, \quad & t \in J,\ x \in \Omega, \\
\partial_t\vartheta + \lambda'(\psi)\partial_t\psi - \Delta\vartheta = f_2, \quad & t \in J,\ x \in \Omega, \\
\partial_\nu\mu = g_1, \quad & t \in J,\ x \in \Gamma, \\
\partial_t\psi - \sigma_s\Delta_\Gamma\psi + \gamma\partial_\nu\psi + \kappa\psi = g_2, \quad & t \in J,\ x \in \Gamma, \\
\alpha\vartheta + \partial_\nu\vartheta = g_3, \quad & t \in J,\ x \in \Gamma, \\
\psi(0) = \psi_0, \quad & t = 0,\ x \in \Omega, \\
\vartheta(0) = \vartheta_0, \quad & t = 0,\ x \in \Omega,
\end{aligned}
\tag{1.5}
$$

where f_i,g_j are given functions in appropriate function spaces to be defined later. Furthermore the initial functions ψ_0, ϑ_0 are given and $\sigma_s, \gamma > 0$, $\alpha, \kappa \geq 0$. Typically one has $\lambda(\psi) = \lambda_0(\psi - \psi_c)$ and $\Phi(\psi) = \Phi_0(\frac{1}{4}\psi^4 - \frac{1}{2}\psi^2)$, with $\lambda_0, \Phi_0 > 0$ and $\psi_c = \text{const}$. In this case Φ is called a *double-well* potential and λ_0 is called *latent heat*; the two distinct minima of Φ characterize the two stable phases during the phase transition.

We are interested in solutions

$$
\psi \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^4(\Omega)) =: Z^1,
$$

$$
\vartheta \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)) =: Z^2,
$$

with

$$
\psi|_\Gamma \in H_p^1(J; W_p^{2-1/p}(\Gamma)) \cap L_p(J; W_p^{4-1/p}(\Gamma)) =: Z_\Gamma^1.
$$

In the following section we will solve the corresponding linear problem to (1.5) to obtain maximal L_p-regularity. Then, in Section 3, the property of maximal L_p-regularity and the contraction mapping principle are applied to overcome the nonlinearities and to obtain local well-posedness of (1.5). Here we employ the technique introduced in the paper of Clément and Li [3]. Section 4 is dedicated to global well-posedness of (1.5). We will make use of some energy estimates and the Gagliardo-Nirenberg inequality to establish global existence again with the help of maximal L_p-regularity. For the sake of readability, the technical parts of the proofs of Theorems 3.3 and 4.2 are given in the appendix. Finally, we investigate the asymptotic behavior of the solutions to (1.5) in Section 5. Here we show relative compactness of the orbits $\psi(\mathbb{R}_+)$ and $\vartheta(\mathbb{R}_+)$ in a suitable energy space and we make use of some differentiability properties of the underlying energy-functionals to (1.5). Lastly the Lojasiewicz-Simon inequality is the key, which leads to convergence of solutions to steady states as time tends to infinity.

2. The linear problem

Before we deal with the linearized version of (1.5), we need some preliminaries. Firstly, we want to set $f_2, g_3, \vartheta_0 = 0$. For this we consider the system

$$\begin{aligned} \partial_t v - \Delta v &= f_2, \quad t \in J,\ x \in \Omega, \\ \alpha v + \partial_\nu v &= g_3, \quad t \in J,\ x \in \Gamma, \\ v(0) &= \vartheta_0, \quad t = 0,\ x \in \Omega. \end{aligned} \tag{2.1}$$

By [4, Theorem 2.1] there is a unique solution $\vartheta_1 \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega))$ of (2.1), provided $f_2 \in L_p(J \times \Omega) =: X$, $\vartheta_0 \in W_p^{2-2/p}(\Omega) =: X_p$,

$$g_3 \in W_p^{1/2-1/2p}(J; L_p(\Gamma)) \cap L_p(J; W_p^{1-1/p}(\Gamma)) =: Y_3,$$

and the compatibility condition $\alpha\vartheta_0 + \partial_\nu \vartheta_0 = g_3|_{t=0}$ is satisfied, whenever $p > 3$. Taking the latter for granted, we may set $f_2, g_3, \vartheta_0 = 0$ in (1.5).

Secondly we want to replace ϑ in $(1.5)_1$ by a term only depending on ψ and some given data. Therefore we will solve the inhomogeneous heat equation

$$\partial_t \vartheta - \Delta \vartheta = -\partial_t \lambda(\psi), \quad \vartheta(0) = 0 \tag{2.2}$$

with homogeneous Robin- or Neumann-boundary conditions. Suppose we already know a solution $(\psi, \vartheta) \in (Z^1 \cap Z_\Gamma^1) \times Z^2$ of (1.5). Assuming that λ' is bounded we have $\partial_t \lambda(\psi) \in L_p(J \times \Omega)$. Let $A_K = -\Delta_K$, $K = R, N$, where R and N stand for Robin- and Neumann-boundary conditions, respectively. By $e^{-A_K t}$ we denote the bounded analytic semigroup, generated by $-A_K$ in $L_p(\Omega)$. The solution ϑ to (2.2) may then be represented by the variation of parameters formula

$$\vartheta(t) = -\int_0^t e^{-A_K(t-s)} \partial_t \lambda(\psi(s))\, ds.$$

Our aim is to split the derivative ∂_t into $\partial_t^{1/2}\partial_t^{1/2}$. But this is only possible if one applies ∂_t to a function with vanishing trace at $t = 0$. Since in general $\lambda(\psi_0) \neq 0$, we insert a function, say $w_0 \in Z^1$, with $w_0(0) = \lambda(\psi_0)$. The existence of such a function w_0 may be seen by solving the problem

$$\begin{aligned} \partial_t w + \Delta^2 w &= 0, \quad t \in J,\ x \in \Omega, \\ \partial_\nu \Delta w &= e^{-\Delta_\Gamma^2 t} \partial_\nu \Delta \lambda(\psi_0), \quad t \in J,\ x \in \Gamma, \\ \partial_\nu w &= e^{-\Delta_\Gamma^2 t} \partial_\nu \lambda(\psi_0), \quad t \in J,\ x \in \Gamma, \\ w(0) &= \lambda(\psi_0), \quad t = 0,\ x \in \Omega, \end{aligned}$$

under the assumption $\psi_0 \in W_p^{4-4/p}(\Omega)$ which is necessary for $\psi \in Z^1$. Then we may write

$$\begin{aligned} \vartheta(t) &= -\int_0^t e^{-A_K(t-s)} \partial_t w_0(s)\, ds - \int_0^t e^{-A_K(t-s)} \partial_t (\lambda(\psi(s)) - w_0(s))\, ds \\ &= \vartheta_2(t) - \partial_t^{1/2}(\partial_t + A_K)^{-1} \partial_t^{1/2} (\lambda(\psi(t)) - w_0(t)), \end{aligned} \tag{2.3}$$

with $\vartheta_2(t) := -\int_0^t e^{-A_K(t-s)}\partial_t w_0(s)\,ds$. As we will see in Section 3, the splitting of the time derivative ∂_t yields a lower order term, compared to $(\partial_t + \Delta^2)\psi$. Thus it remains to solve the problem

$$\begin{aligned} \partial_t\psi + \Delta^2\psi = \Delta\Phi'(\psi) + \Delta(\lambda'(\psi)F(\psi)) - \Delta(\lambda'(\psi)\vartheta^*) + f, &\quad t \in J,\ x \in \Omega,\\ \partial_\nu\Delta\psi = \partial_\nu\Phi'(\psi) + \partial_\nu(\lambda'(\psi)F(\psi)) - \partial_\nu(\lambda'(\psi)\vartheta^*) + g, &\quad t \in J,\ x \in \Gamma,\\ \partial_t\psi - \sigma_s\Delta_\Gamma\psi + \gamma\partial_\nu\psi + \kappa\psi = h, &\quad t \in J,\ x \in \Gamma,\\ \psi(0) = \psi_0, &\quad t = 0,\ x \in \Omega, \end{aligned} \tag{2.4}$$

with $F(\psi) := \partial_t^{1/2}(\partial_t + A_K)^{-1}\partial_t^{1/2}(\lambda(\psi) - w_0)$ and $\vartheta^* = \vartheta_1 + \vartheta_2 \in Z^2$. The corresponding linear system to (2.4) reads as follows

$$\begin{aligned} \partial_t u + \Delta^2 u = f, &\quad t \in J,\ x \in \Omega,\\ \partial_\nu\Delta u = g, &\quad t \in J,\ x \in \Gamma,\\ \partial_t u - \sigma_s\Delta_\Gamma u + \gamma\partial_\nu u + \kappa u = h, &\quad t \in J,\ x \in \Gamma,\\ u(0) = u_0, &\quad t = 0,\ x \in \Omega. \end{aligned} \tag{2.5}$$

Here is the main result on maximal L_p-regularity of (2.5).

Theorem 2.1. *Let $n \in \mathbb{N}$, $1 < p < \infty$, $p \neq 3, 5$ and let $\sigma_s, \gamma > 0$ and $\kappa \geq 0$ be constants. Suppose $\Omega \subset \mathbb{R}^n$ is bounded open with compact boundary $\Gamma = \partial\Omega \in C^4$ and let $J_0 = [0, T_0]$. Then there is a unique solution u of (2.5) such that*

$$u \in H_p^1(J_0; L_p(\Omega)) \cap L_p(J_0; H_p^4(\Omega)) = Z^1,$$

with

$$u|_\Gamma \in H_p^1(J_0; W_p^{2-1/p}(\Gamma)) \cap L_p(J_0; W_p^{4-1/p}(\Gamma)) = Z_\Gamma^1,$$

if and only if the data are subject to the following conditions.

(i) $f \in L_p(J_0 \times \Omega) = X$,
(ii) $g \in W_p^{1/4-1/4p}(J_0; L_p(\Gamma)) \cap L_p(J_0; W_p^{1-1/p}(\Gamma)) =: Y_1$,
(iii) $h \in L_p(J_0; W_p^{2-1/p}(\Gamma)) =: Y_2$,
(iv) $u_0 \in \{w \in B_{pp}^{4-4/p}(\Omega) : w|_\Gamma \in B_{pp}^{4-3/p}(\Gamma)\} =: X_{p,\Gamma}$,
(v) $\partial_\nu\Delta u_0 = g|_{t=0}$, *if* $p > 5$.

Proof. This theorem is a special case of [8, Theorem 2.1]. □

Remark. By B_{pq}^s we denote the Besov-spaces as defined, e.g., in Triebel [10]. Observe that $B_{pp}^s = W_p^s$ for all $s \notin \mathbb{N}$. Thus in (iv) of Theorem 2.1 we could have written W instead of B on the expense of excluding the values of $p \neq 2$ where $4 - 4/p$ or $4 - 3/p$ is integer.

3. Local well-posedness

In this section we will apply the contraction mapping principle to overcome the nonlinearities in

$$\begin{aligned} \partial_t u + \Delta^2 u &= \Delta G(u) + f, && t \in J,\ x \in \Omega, \\ \partial_\nu \Delta u &= \partial_\nu G(u) + g, && t \in J,\ x \in \Gamma, \\ \partial_t u - \sigma_s \Delta_\Gamma u + \gamma \partial_\nu u + \kappa u &= h, && t \in J,\ x \in \Gamma, \\ u(0) &= \psi_0, && t = 0,\ x \in \Omega, \end{aligned} \tag{3.1}$$

where

$$G(u) := \Phi'(u) + \lambda'(u) F(u) - \lambda'(u)\vartheta^*.$$

For that purpose let $\psi_0 \in X_{p,\Gamma}$, $f \in X$, $g \in Y_1$ and $h \in Y_2$, as well as $\vartheta_0 \in X_p$ be given, such that the compatibility condition

$$\partial_\nu \Delta \psi_0 = \Phi'(\psi_0) - \lambda'(\psi_0)\vartheta_0 + g|_{t=0}, \quad \text{if } p > 5,$$

is satisfied. We furthermore assume that $\lambda, \Phi \in C^{4-}(\mathbb{R})$ and we will use the embeddings

$$H^1_p(J; L_p(\Omega)) \cap L_p(J; H^4_p(\Omega)) \hookrightarrow C(J; W^{4-4/p}_p(\Omega)) \hookrightarrow C(J \times \overline{\Omega}), \tag{3.2}$$

valid for $p > n/4 + 1$. For $[0,T] \subset [0,T_0]$ we define

$$\mathbb{E}_1 := \{ w \in Z^1(T) : w|_\Gamma \in Z^1_\Gamma(T) \}, \quad \text{and} \quad {}_0\mathbb{E}_1 := \{ w \in \mathbb{E}_1 : w|_{t=0} = 0 \}$$

and

$$\mathbb{E}_0 := X(T) \times Y_1(T) \times Y_2(T), \quad \text{and} \quad {}_0\mathbb{E}_0 := \{ (f,g,h) \in \mathbb{E}_0 : g|_{t=0} = 0 \}.$$

The spaces $\mathbb{E}_1$ and $\mathbb{E}_0$ are endowed with canonical norms $|\cdot|_1$ and $|\cdot|_0$, respectively. Let furthermore $A := -\Delta_\Gamma$. By Theorem 2.1 there is a unique solution $u^* \in \mathbb{E}_1$ of the linear system

$$\begin{aligned} \partial_t u + \Delta^2 u &= f, && t \in J,\ x \in \Omega, \\ \partial_\nu \Delta u &= g - e^{-A^2 t} g_0, && t \in J,\ x \in \Gamma, \\ \partial_t u - \sigma \Delta_\Gamma u + \gamma \partial_\nu u + \kappa u &= h, && t \in J,\ x \in \Gamma, \\ u(0) &= \psi_0, && t = 0,\ x \in \Omega, \end{aligned} \tag{3.3}$$

where $g_0 = 0$, if $p < 5$ and $g_0 = g|_{t=0} - \partial_\nu \Delta \psi_0$, if $p > 5$. We define a linear operator $\mathbb{L} : \mathbb{E}_1 \to \mathbb{E}_0$ by

$$\mathbb{L} w = \begin{bmatrix} \partial_t w + \Delta^2 w \\ \partial_\nu \Delta w \\ \partial_t w - \sigma_s \Delta_\Gamma w + \mu \partial_\nu w + \kappa w \end{bmatrix}.$$

Consider $\mathbb{L}$ as an operator from ${}_0\mathbb{E}_1$ to ${}_0\mathbb{E}_0$. Then by Theorem 2.1, $\mathbb{L}$ is bounded and bijective, i.e., an isomorphism. Hence, by the open-mapping theorem, $\mathbb{L}$ is invertible with bounded inverse.

Next we define a nonlinear mapping $\tilde{G} : \mathbb{E}_1 \times {}_0\mathbb{E}_0 \to {}_0\mathbb{E}_0$ by means of

$$\tilde{G}(u^*, w) = \begin{bmatrix} \Delta G(u^* + w) \\ \partial_\nu G(u^* + w) - g_1 \\ 0 \end{bmatrix},$$

where $g_1 = 0$, if $p < 5$ and $g_1 = e^{-A^2 t}\partial_\nu G(u^*)|_{t=0}$ if $p > 5$. We will show in a subsequent proposition that the range of $\tilde{G}$ is indeed a subset of ${}_0\mathbb{E}_0$. For the moment, assume that this result is already at our disposal. It is then obvious that $u := u^* + w$ is a solution of (3.1) if and only if $\mathbb{L}w = \tilde{G}(u^*, w)$ or equivalently $w = \mathbb{L}^{-1}\tilde{G}(u^*, w)$. In addition we define a ball $\mathbb{B}_R \subset {}_0\mathbb{E}_1$ by

$$\mathbb{B}_R := \mathbb{B}_R(0) := \{w \in_0 \mathbb{E}_1 \ : \ |w|_1 \le R\}, \ R \in (0, 1],$$

and an operator $S : \mathbb{B}_R \to {}_0\mathbb{E}_1$ by $Sw = \mathbb{L}^{-1}\tilde{G}(u^*, w)$. In order to apply the contraction mapping principle we have to ensure that S is a self-mapping, i.e., $S\mathbb{B}_R \subset \mathbb{B}_R$ and that S defines a strict contraction on $\mathbb{B}_R$, i.e., there exists a number $\beta < 1$ with

$$|Sw - S\bar{w}|_1 \le \beta|w - \bar{w}|_1, \quad w, \bar{w} \in \mathbb{B}_R.$$

We need some preliminaries to prove these properties. First we observe that all functions belonging to $\mathbb{B}_R$ are uniformly bounded on $J \times \Omega$. Indeed, by (3.2) it holds that

$$|w|_\infty \le M|w|_{Z^1} \le M|w|_1 \le MR \le M,$$

with a constant $M > 0$, independent of T, since $w|_{t=0} = 0$ for all $w \in \mathbb{B}_R$.

For all forthcoming considerations we define the shifted ball $\mathbb{B}_R(u^*) \subset \mathbb{E}_1$ by means of

$$\mathbb{B}_R(u^*) := \{w \in \mathbb{E}_1 : w = \tilde{w} + u^*, \ \tilde{w} \in \mathbb{B}_R\}.$$

Note that all functions $w \in \mathbb{B}_R(u^*)$ are uniformly bounded, too. In the sequel we will also make use of the well-known estimate

$$|fg|_{H^s_p(L_p)} \le C(|f|_{L_{\sigma_1' p}(L_{r_1' p})}|g|_{H^s_{\sigma_1 p}(L_{r_1 p})} + |g|_{L_{\sigma_2' p}(L_{r_2' p})}|f|_{H^s_{\sigma_2 p}(L_{r_2 p})}), \qquad (3.4)$$

where $1/\sigma_i + 1/\sigma_i' = 1/r_i + 1/r_i' = 1$, $i = 1, 2$, $s \in [0, 1]$. Indeed, this is a consequence of the definition of the spaces H^s_p and Hölders inequality (cf. also [10]). Next, by [12, Lemma 6.2.3] there is a function $\mu(T) > 0$, with $\mu(T) \to 0$ as $T \to 0$, such that

$$|f(u) - f(v)|_{H^s_p(L_p)} \le \mu(T)(|u - v|_{H^{s_0}_p(L_p)} + |u - v|_\infty), \quad 0 < s < s_0 < 1, \quad (3.5)$$

for every $f \in C^{1-}(\mathbb{R})$ and all $u, v \in \mathbb{B}_R(u^*)$. Now we show that $F(w)$, $w \in \mathbb{B}_R(u^*)$, represents a lower order term in (3.1). By the mixed-derivative theorem we obtain

$$w_0 \in Z^1 \hookrightarrow H^{3/4}_p(J; H^1_p(\Omega)) \hookrightarrow H^s_p(J; H^1_p(\Omega)),$$

for every $s \in (0, 3/4)$. By (3.4) we see that $\lambda(w) \in H^s_p(J; H^1_p(\Omega))$, $s \in (0, 3/4)$, too. Thus $(\lambda(w) - w_0) \in {}_0H^s_p(J; H^1_p(\Omega))$, and for $1/2 \le s < 3/4$ and every

$\eta \in {}_0H^s_p(J; H^1_p(\Omega))$ we obtain

$$(\partial_t + A_K)^{-1}\partial_t^{1/2}\eta \in {}_0H^{s+1/2}_p(J; H^1_p(\Omega)) \cap {}_0H^{s-1/2}_p(J; H^3_p(\Omega)) \hookrightarrow {}_0H^{s+\theta-1/2}_p(J; H^{3-2\theta}_p(\Omega)),$$

for each $\theta \in [\frac{1}{2}, 1]$, where the latter embedding is due to the mixed-derivative theorem. Finally it holds

$$\partial_t^{1/2}(\partial_t + A_K)^{-1}\partial_t^{1/2} : {}_0H^s_p(J; H^1_p(\Omega)) \to {}_0H^{s+\theta-1}_p(J; H^{3-2\theta}_p(\Omega)), \quad \theta \in [\frac{1}{2}, 1]. \tag{3.6}$$

The following proposition shows that the Lipschitz property of λ carries over to F.

Proposition 3.1. *Let $p > n/4 + 1$, $\lambda \in C^{2-}(\mathbb{R})$ and $J = [0, T] \subset [0, T_0]$. Then there exists a function $\mu(T) > 0$, with $\mu(T) \to 0$ as $T \to 0$, such that for every $s \in [\frac{1}{2}, \frac{3}{4})$, and all $u, v \in \mathbb{B}_R(u^*)$ the estimate*

$$|F(u) - F(v)|_{H^{s-1/2}_p(H^2_p)} + |\nabla F(u) - \nabla F(v)|_{H^{s-1/2}_p(H^1_p)} + |\Delta F(u) - \Delta F(v)|_{H^{s-1/2}_p(L_p)} \leq \mu(T)|u - v|_1$$

is valid.

Proof. By (3.6) it suffices to show that

$$|\lambda(u) - \lambda(v)|_{H^s_p(H^1_p)} \leq \mu(T)|u - v|_1.$$

Obviously $|\lambda(u)-\lambda(v)|_{H^s_p(H^1_p)} \leq C(|\lambda(u)-\lambda(v)|_{H^s_p(L_p)}+|\nabla u\lambda'(u)-\nabla v\lambda'(v)|_{H^s_p(L_p)})$ and

$$|\nabla u\lambda'(u)-\nabla v\lambda'(v)|_{H^s_p(L_p)} \leq C(|\nabla u(\lambda'(u)-\lambda'(v))|_{H^s_p(L_p)}+|\lambda'(v)(\nabla u-\nabla v)|_{H^s_p(L_p)}).$$

Now (3.4) yields

$$|\nabla u(\lambda'(u) - \lambda'(v))|_{H^s_p(L_p)} \leq C(T_0)\Big(|\nabla u|_{L_{r'p}(L_{r'p})}|\lambda'(u) - \lambda'(v)|_{H^s_{rp}(L_{rp})} + T^{1/\sigma'}|\lambda'(u) - \lambda'(v)|_\infty |\nabla u|_{H^s_{\sigma p}(L_{\sigma p})}\Big),$$

as well as

$$|\lambda'(v)(\nabla u - \nabla v)|_{H^s_p(L_p)} \leq C(T_0)\Big(|\nabla u - \nabla v|_{L_{r'p}(L_{r'p})}|\lambda'(v)|_{H^s_{rp}(L_{rp})} + T^{1/\sigma'}|\nabla u - \nabla v|_{H^s_{\sigma p}(L_{\sigma p})}\Big).$$

Again by (3.4) we see that $|\lambda'(v)|_{H^s_{rp}(L_{rp})} \leq T^{1/\rho' p}|\lambda'(v)|_{H^s_{r\rho p}(L_{r\rho p})}$. Observe that the embedding $Z^1 \hookrightarrow H^s_{rp}(L_{rp})$ holds, whenever $s \leq 1/r$. To meet this requirement we set $r = 4/3$, i.e., $r' = 4$. Hence for sufficiently small $\rho > 1$ and by (3.5) we obtain the desired estimate. □

The next proposition collects all the facts we need to show the desired properties of the operator S defined above.

Proposition 3.2. *Let $p > n/4 + 1$, $\lambda, \Phi \in C^{4-}(\mathbb{R})$ and $J = [0, T] \subset [0, T_0]$. Then there exists a constant $C > 0$, independent of T, and functions $\mu_j = \mu_j(T)$ with $\mu_j(T) \to 0$ as $T \to 0$, $j = 1, \dots, 4$, such that for all $u, v \in \mathbb{B}_R(u^*)$ the following statements hold.*

(i) $|\Delta\Phi'(u) - \Delta\Phi'(v)|_p \le \mu_1(T)|u - v|_1$,
(ii) $|(\Delta(\lambda'(u)F(u)) - \Delta(\lambda'(v)F(v))|_p \le \mu_2(T)|u - v|_1$,
(iii) $|\partial_\nu\Phi'(u) - \partial_\nu\Phi'(v)|_{Y_1(T)} \le \mu_3(T)|u - v|_1$,
(iv) $|\partial_\nu(\lambda'(u)F(u)) - \partial_\nu(\lambda'(v)F(v))|_{Y_1(T)} \le \mu_4(T)|u - v|_1$.

For each $\eta \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega))$ we have

(v) $|(\Delta(\lambda'(u)\eta) - \Delta(\lambda'(v)\eta)|_p \le C|\eta|_{Z^2}|u - v|_1$,
(vi) $|(\partial_\nu(\lambda'(u)\eta) - \partial_\nu(\lambda'(v)\eta)|_{Y_1(T)} \le C|\eta|_{Z^2}|u - v|_1$.

The proof is given in the Appendix.

Note that we have also shown that $\Delta G(w) \in X(T)$ and $\partial_\nu G(w) \in Y_1(T)$ for each $w \in \mathbb{B}_R(u^*)$, where G was defined in (3.1). Thus the operator $S : \mathbb{B}_R \to {}_0\mathbb{E}_1$ is well defined. With our previous considerations, it is now easy to verify the self-mapping property as well as the strict contraction property of S. Let $w \in \mathbb{B}_R$. Then we obtain

$$\begin{aligned}|Sw|_1 &= |\mathbb{L}^{-1}\tilde{G}(u^*, w)|_1 \le |\mathbb{L}^{-1}||\tilde{G}(u^*, w)|_0\\ &\le C(|\tilde{G}(u^*, w) - \tilde{G}(u^*, 0)|_0 + |\tilde{G}(u^*, 0)|_0)\\ &\le C(|\Delta G(u^* + w) - \Delta G(u^*)|_{X(T)} + |\partial_\nu(G(u^* + w) - G(u^*))|_{Y_1(T)}\\ &\qquad + |\Delta G(u^*)|_{X(T)} + |\partial_\nu G(u^*)|_{Y_1(T)} + |g_1|_{Y_1(T)}.\end{aligned}$$

By Proposition 3.2 there exists a function $\mu(T)$, with $\mu(T) \to 0$ as $T \to 0$, such that

$$|\Delta G(u^* + w) - \Delta G(u^*)|_{X(T)} + |\partial_\nu(G(u^* + w) - G(u^*))|_{Y_1(T)} \le \mu(T)|w|_1 \le \mu(T)R,$$

since $w + u^* \in \mathbb{B}_R(u^*)$. Thus we see that $|Sw|_1 \le R$, if $T > 0$ is sufficiently small. We remark that g_1 and $G(u^*)$ are fixed functions, hence $|g_1|_{Y_1(T)}$, $|\Delta G(u^*)|_{X(T)}$, $|\partial_\nu G(u^*)|_{Y_1(T)} \to 0$ as $T \to 0$. This shows that $S\mathbb{B}_R \subset \mathbb{B}_R$. Furthermore for all $w, \bar{w} \in \mathbb{B}_R$ we have

$$\begin{aligned}|Sw - S\bar{w}|_1 &= |\mathbb{L}^{-1}(\tilde{G}(u^*, w) - \tilde{G}(u^*, \bar{w}))|_1 \le |\mathbb{L}^{-1}||\tilde{G}(u^*, w) - \tilde{G}(u^*, \bar{w})|_0\\ &\le C(|\Delta G(u^* + w) - \Delta G(u^* + \bar{w})|_{X(T)} + |\partial_\nu G(u^* + w) - \partial_\nu G(u^* + \bar{w})|_{Y_1(T)}).\end{aligned}$$

It is a consequence of Proposition 3.2 that

$$|\Delta G(u^*+w) - \Delta G(u^*+\bar{w})|_{X(T)} + |\partial_\nu G(u^*+w) - \partial_\nu G(u^*+\bar{w})|_{Y_1(T)} \le \mu(T)|w - \bar{w}|_1,$$

hence $S : \mathbb{B}_R \to \mathbb{B}_R$ is a strict contraction, if $T > 0$ is sufficiently small. Thus the contraction-mapping principle yields a unique fixed-point w^* of S, i.e., a solution $\psi \in Z^1(T) \cap Z^1_\Gamma(T)$ of (2.4), which depends continuously on the given data $f \in X$, $g \in Y_1$, $h \in Y_2$ and $\psi_0 \in X_{p,\Gamma}$. Since $\partial_t\lambda(\psi) = \partial_t\psi\lambda'(\psi) \in L_p(J \times \Omega)$, there is a unique solution $\vartheta \in Z^2(T)$ of

$$\begin{aligned} \partial_t v - \Delta v &= -\partial_t \lambda(\psi) + f_2, \quad t \in J,\ x \in \Omega, \\ \alpha v + \partial_\nu v &= g_3, \quad t \in J,\ x \in \Gamma, \\ v(0) &= \vartheta_0, \quad t = 0,\ x \in \Omega. \end{aligned}$$

Finally we see that $(\psi, \vartheta) \in (Z^1(T) \cap Z^1_\Gamma(T)) \times Z^2(T)$ is the unique solution of (1.5) on the interval $[0, T]$. We summarize these considerations in

Theorem 3.3. *Let $1 < p < \infty$, $p > n/4 + 1$, $p \neq 3, 5$ and let $\sigma_s, \gamma > 0$, $\alpha, \kappa \geq 0$ be constants. Assume furthermore that $\lambda, \Phi \in C^{4-}(\mathbb{R})$. Then there exists an interval $J = [0, T] \subset [0, T_0]$ and a unique solution (ψ, ϑ) of* (1.5) *on J, with*

$$\begin{aligned} \psi &\in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^4_p(\Omega)) = Z^1(T), \\ \vartheta &\in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega)) = Z^2(T), \end{aligned}$$

and

$$\psi|_\Gamma \in H^1_p(J; W^{2-1/p}_p(\Gamma)) \cap L_p(J; W^{4-1/p}_p(\Gamma)) = Z^1_\Gamma(T),$$

provided the data are subject to the following conditions.

(i) $f_1, f_2 \in L_p(J \times \Omega) = X$,
(ii) $g_1 \in W^{1/4-1/4p}_p(J; L_p(\Gamma)) \cap L_p(J; W^{1-1/p}_p(\Gamma)) = Y_1$,
(iii) $g_2 \in L_p(J; W^{2-1/p}_p(\Gamma)) = Y_2$,
(iv) $g_3 \in W^{1/2-1/2p}_p(J; L_p(\Gamma)) \cap L_p(J; W^{1-1/p}_p(\Gamma)) = Y_3$,
(v) $\psi_0 \in \{u \in B^{4-4/p}_{pp}(\Omega) : u|_\Gamma \in B^{4-3/p}_{pp}(\Gamma)\} = X_{p,\Gamma}$,
(vi) $\vartheta_0 \in W^{2-2/p}_p(\Omega) = X_p$,
(vii) $\partial_\nu \Delta \psi_0 = -\Phi'(\psi_0) + \lambda'(\psi_0)\vartheta_0 + g_1|_{t=0}$, *if* $p > 5$,
(viii) $\alpha\vartheta_0 + \partial_\nu \vartheta_0 = g_3|_{t=0}$, *if* $p > 3$.

The solution depends continuously on the given data and if the data are independent of t, the map $(\psi_0, \vartheta_0) \mapsto (\psi, \vartheta)$ defines a local semiflow on the natural phase manifold $\mathcal{M} \subset X_{p,\Gamma} \times X_p$, defined by (vii) *and* (viii).

4. Global well-posedness

Throughout this section we assume that $n \leq 3$ and that the potential Φ satisfies the growth conditions

$$\Phi(s) \geq -\frac{\eta}{2}s^2 - c_0, \quad c_0 > 0,\ s \in \mathbb{R}, \tag{4.1}$$

where $\eta < \lambda_1$, with λ_1 being the smallest nontrivial eigenvalue of the negative Laplacian on Ω with Neumann boundary conditions,

$$|\Phi'(s)| \leq (c_1\Phi(s) + c_2 s^2 + c_3)^\theta, \quad c_i > 0,\ \theta \in (0,1),\ s \in \mathbb{R}, \tag{4.2}$$

and

$$|\Phi'''(s)| \leq C(1 + |s|^\beta), \quad s \in \mathbb{R}, \tag{4.3}$$

with $\beta < 3$ in case $n = 3$. Furthermore let $\lambda', \lambda'', \lambda''' \in L_\infty(\mathbb{R})$.

Remark. (i) The conditions (4.1)–(4.3) are certainly fulfilled, if Φ is a polynomial of degree $2m$, $m \in \mathbb{N}$, $m < 3$. Then we may set $\theta = 1 - 1/2m$.

(ii) As we will see we may omit (4.2) if $f_1 = g_1 = 0$.

A successive application of Theorem 3.3 yields a maximal time interval $J_{\max} = [0, T_{\max}) \subset [0, T_0]$ for the solution $\psi \in Z^1 \cap Z^1_\Gamma$ of (3.1). If $T_{\max} < T_0$, this interval is characterized by the following two equivalent conditions

$$\lim_{t \to T_{\max}} \psi(t) \quad \text{does not exist in } X_{p,\Gamma}$$

and

$$|\psi|_{Z^1(T_{\max})} + |\psi_{|\Gamma}|_{Z^1_\Gamma(T_{\max})} = \infty.$$

First of all, we need some a priori estimates for ψ. We multiply $(1.5)_1$ by μ and $(1.5)_2$ by ϑ. Integration by parts and the boundary conditions $(1.5)_{3,4,5}$ lead to the energy-equation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(|\nabla\psi|_2^2 + |\vartheta|_2^2 + \frac{\sigma_s}{\gamma}|\nabla_\Gamma\psi|_{2,\Gamma}^2 + \frac{\kappa}{\gamma}|\psi|_{2,\Gamma}^2 + 2\int_\Omega \Phi(\psi)\Big)&\\ + |\nabla\mu|_2^2 + \frac{1}{\gamma}|\partial_t\psi|_{2,\Gamma}^2 + |\nabla\vartheta|_2^2 + \alpha|\vartheta|_{2,\Gamma}^2 &\quad (4.4)\\ = \int_\Omega (f_1\mu + f_2\vartheta) + \int_\Gamma (g_1\mu + \frac{1}{\gamma}g_2\partial_t\psi + g_3\vartheta).&\end{aligned}$$

The Poincaré-Wirtinger inequality as well as the embedding $H^1_2(\Omega) \hookrightarrow L_2(\Gamma)$ imply

$$\int_\Omega f_1\mu \le C|f_1|_2(|\nabla\mu|_2 + |\int_\Omega \mu|) \quad \text{and} \quad \int_\Gamma g_1\mu \le C|g_1|_{2,\Gamma}(|\nabla\mu|_2 + |\int_\Omega \mu|). \quad (4.5)$$

By the definition of the chemical potential μ, by the divergence theorem and by the dynamic boundary condition $(1.5)_4$ we have

$$\int_\Omega \mu = \int_\Omega (\Phi'(\psi) - \lambda'(\psi)\vartheta) + \frac{1}{\gamma}\int_\Gamma (\partial_t\psi + \kappa\psi - g_2),$$

hence using (4.2) and the Cauchy-Schwarz inequality

$$|\int_\Omega \mu| \le C(|\vartheta|_2 + |\partial_t\psi|_{2,\Gamma} + |\psi|_{2,\Gamma} + |g_2|_{2,\Gamma}) + \int_\Omega (c_1\Phi(\psi) + c_2|\psi|^2 + c_3)^\theta.$$

For simplicity, we set

$$E(\psi,\vartheta) := \frac{1}{2}|\nabla\psi|_2^2 + \frac{1}{2}|\vartheta|_2^2 + \frac{1}{2}\frac{\sigma_s}{\gamma}|\nabla_\Gamma\psi|_{2,\Gamma}^2 + \frac{1}{2}\frac{\kappa}{\gamma}|\psi|_{2,\Gamma}^2 + \int_\Omega \Phi(\psi).$$

From (4.4), (4.5) and by Hölder's inequality, Young's inequality as well as by the Poincaré-Wirtinger inequality we obtain the estimate

$$\begin{aligned}&\frac{d}{dt}\,E(\psi,\vartheta) + C(|\nabla\mu|_2^2 + |\partial_t\psi|_{2,\Gamma}^2 + |\nabla\vartheta|_2^2 + \alpha|\vartheta|_{2,\Gamma}^2)\\ &\quad \le C_1E(\psi,\vartheta) + C_2(|f_1|_2^q + |f_2|_2^2 + |g_1|_{2,\Gamma}^q + |g_2|_{2,\Gamma}^2 + |g_3|_{2,\Gamma}^2 + 1),\end{aligned}$$

where $q := \max\{2, \frac{1}{1-\theta}\}$. Observe that the functional E is bounded from below. Indeed, by (4.1) we obtain

$$E(\psi(t),\vartheta(t)) \geq \frac{1}{2}(|\nabla\psi|_2^2 - \eta|\psi|_2^2) - c_0|\Omega| \geq \frac{\lambda_1-\eta}{2\lambda_1}|\nabla\psi|_2^2 - c \geq -c, \quad c>0, \tag{4.6}$$

where we used again the Poincaré inequality, since

$$|\int_\Omega \psi(t)| \leq \int_\Omega |\psi_0| + |f_1|_{L_1(J\times\Omega)} + |g_1|_{L_1(J\times\Gamma)}. \tag{4.7}$$

Then Gronwall's lemma yields the estimate

$$E(\psi,\vartheta) \leq C\left(E(\psi_0,\vartheta_0) + \int_0^{T_{\max}}(|f_1|_2^q + |f_2|_2^2 + |g_1|_{2,\Gamma}^q + |g_2|_{2,\Gamma}^2 + |g_3|_{2,\Gamma}^2 + 1)\right),$$

and by (4.7) we obtain among other things the a priori estimate

$$\psi \in L_\infty([0,T_{\max}]; H_2^1(\Omega)).$$

The following lemma is the key to obtain global existence.

Lemma 4.1. *Suppose $p \geq 2$, $n \leq 3$ and let $\psi \in Z^1(T)$ be the solution of (3.1). Then there exist constants $m, C > 0$ and $\delta \in (0,1)$, independent of $T > 0$, such that*

$$|\Delta G(\psi)|_{X(T)} + |\partial_\nu G(\psi)|_{Y_1(T)} \leq C(1 + |\psi|_{Z^1(T)}^\delta |\psi|_{L_\infty(J;H_2^1(\Omega))}^m).$$

The proof is given in the Appendix.

Observe that by maximal L_p-regularity the estimate

$$\begin{aligned}&|\psi|_{Z^1(T)} + |\psi|_{Z_\Gamma^1(T)}\\ &\quad\leq M(|\Delta G(\psi)|_{X(T)} + |\partial_\nu G(\psi)|_{Y_1(T)} + |f|_X + |g|_{Y_1} + |h|_{Y_2} + |\psi_0|_{X_{p,\Gamma}}),\end{aligned}$$

for the solution ψ of (3.1) is valid, with a constant $M = M(T_0) > 0$. Then it follows from Lemma 4.1 that

$$|\psi|_{Z^1(T)} + |\psi_{|\Gamma}|_{Z_\Gamma^1(T)} \leq M(1 + |\psi|_{Z^1(T)}^\delta),$$

hence $|\psi|_{Z^1(T)}$ is bounded. This in turn yields the boundedness of $|\psi_{|\Gamma}|_{Z_\Gamma^1(T)}$ and therefore global existence of the solution (ψ,ϑ) of (1.5), since ϑ solves the heat-equation $\partial_t\vartheta - \Delta\vartheta = -\partial_t\lambda(\psi)$, subject to homogeneous boundary and initial conditions.

We summarize our previous considerations in

Theorem 4.2. *Let $1 < p < \infty$, $p \geq 2$, $p \neq 3, 5$, $\theta \in (0,1)$, $q = \max\{2, \frac{1}{1-\theta}\}$ and $J_0 = [0, T_0]$. Assume furthermore that $\lambda', \lambda'', \lambda''' \in L_\infty(\mathbb{R})$ and let Φ satisfy (4.1)–(4.3). Then there exists a unique global solution (ψ,ϑ) of (1.5) on J_0, with*

$$\psi \in H_p^1(J_0; L_p(\Omega)) \cap L_p(J_0; H_p^4(\Omega)), \quad \vartheta \in H_p^1(J_0; L_p(\Omega)) \cap L_p(J_0; H_p^2(\Omega)),$$

and

$$\psi|_\Gamma \in H_p^1(J_0; W_p^{2-1/p}(\Gamma)) \cap L_p(J_0; W_p^{4-1/p}(\Gamma)),$$

if the data are subject to the following conditions.

(i) $f_1, f_2 \in L_p(J_0 \times \Omega)$, $f_1 \in L_q(J_0; L_2(\Omega))$,
(ii) $g_1 \in W_p^{1/4-1/4p}(J_0; L_p(\Gamma)) \cap L_p(J_0; W_p^{1-1/p}(\Gamma)) \cap L_q(J_0; L_2(\Gamma))$,
(iii) $g_2 \in L_p(J_0; W_p^{2-1/p}(\Gamma))$,
(iv) $g_3 \in W_p^{1/2-1/2p}(J_0; L_p(\Gamma)) \cap L_p(J_0; W_p^{1-1/p}(\Gamma))$,
(v) $\psi_0 \in \{u \in B_{pp}^{4-4/p}(\Omega) : u|_\Gamma \in B_{pp}^{4-3/p}(\Gamma)\} = X_{p,\Gamma}$,
(vi) $\vartheta_0 \in W_p^{2-2/p}(\Omega) = X_p$,
(vii) $\partial_\nu \Delta\psi_0 = -\Phi'(\psi_0) + \lambda'(\psi_0)\vartheta_0 + g_1|_{t=0}$, *if* $p > 5$,
(viii) $\alpha\vartheta_0 + \partial_\nu\vartheta_0 = g_3|_{t=0}$, *if* $p > 3$.

The solution depends continuously on the given data and if the data are independent of t, *the map* $(\psi_0, \vartheta_0) \mapsto (\psi, \vartheta)$ *defines a global semiflow on the natural phase manifold* $\mathcal{M} \subset X_{p,\Gamma} \times X_p$, *defined by* (vii) *and* (viii).

5. Asymptotic behavior

In this section we study the asymptotic behavior of a solution of the system

$$\begin{aligned} \partial_t\psi - \Delta\mu = 0, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, &\quad t > 0,\ x \in \Omega,\\ \partial_t\vartheta + \lambda'(\psi)\partial_t\psi - \Delta\vartheta = 0, &\quad t > 0,\ x \in \Omega,\\ \partial_\nu\mu = 0, &\quad t > 0,\ x \in \Gamma,\\ \alpha\vartheta + \partial_\nu\vartheta = 0, &\quad t > 0,\ x \in \Gamma,\\ \partial_t\psi - \sigma_s\Delta_\Gamma\psi + \gamma\partial_\nu\psi + \kappa(\psi - h) = 0, &\quad t > 0,\ x \in \Gamma,\\ \psi(0) = \psi_0, &\quad t = 0,\ x \in \Omega,\\ \vartheta(0) = \vartheta_0, &\quad t = 0,\ x \in \Omega. \end{aligned} \tag{5.1}$$

For the forthcoming considerations, we need the following assumptions. Let λ', λ'', $\lambda''' \in L_\infty(\mathbb{R})$ and let Φ satisfy (4.1) as well as (4.3).

The main tool will be the *Lojasiewicz-Simon inequality* (see Proposition 5.3), which leads to the convergence result. We define two energy-functionals $E_K(u, v)$ by means of

$$\begin{aligned} E_N(u,v) = \frac{1}{2}\left(\int_\Omega |\nabla u|^2 + \int_\Omega |v + \overline{\lambda(u)}|^2 + \frac{\sigma_s}{\gamma}\int_\Gamma |\nabla_\Gamma u|^2 + \frac{\kappa}{\gamma}\int_\Gamma |u|^2\right)\\ + \int_\Omega \Phi(u) + \frac{|\Omega|}{2}\,(\overline{\lambda(u)})^2, \end{aligned}$$

and

$$E_R(u,v) = \frac{1}{2}\left(\int_\Omega |\nabla u|^2 + \int_\Omega |v|^2 + \frac{\sigma_s}{\gamma}\int_\Gamma |\nabla_\Gamma u|^2 + \frac{\kappa}{\gamma}\int_\Gamma |u|^2\right) + \int_\Omega \Phi(u),$$

in case of Neumann- or Robin-boundary conditions, respectively. Here we use the abbreviation

$$\overline{w} = \frac{1}{|\Omega|}\int_\Omega w(x)dx$$

for the mean value of a function $w \in L_1(\Omega)$. A suitable energy space both for E_N and E_R will be

$$W := \{(u,v) \in H_2^1(\Omega) \times L_2(\Omega) : u|_\Gamma \in H_2^1(\Gamma),\ \int_\Omega u = 0\}.$$

Note that the condition $\int_\Omega u = 0$ is compatible with our system. This might be seen by integrating $(5.1)_1$ and invoking the boundary condition $(5.1)_3$. We obtain $\int_\Omega \psi = \int_\Omega \psi_0$. If we replace the solution ψ by $\tilde{\psi} = \psi - c$, with $c = \frac{1}{|\Omega|}\int_\Omega \psi_0$ and set $h = c$ in (5.1), we see, that $\tilde{\psi}$ satisfies the corresponding homogeneous system to (5.1) (i.e., $h = 0$), if $\Phi(s)$ and $\lambda(s)$ are replaced by $\Phi_1(s) := \Phi(s+c)$ and $\lambda_1(s) := \lambda(s+c)$, respectively. In a similar way we can achieve that in case $\alpha = 0$ we have $\int_\Omega(\vartheta + \lambda(\psi)) = 0$. Indeed this follows by a shift of λ, i.e., $\tilde{\lambda}(s) := \lambda(s) - d$, where $d = \frac{1}{|\Omega|}\int_\Omega(\vartheta_0 + \lambda(\psi_0))$.

With the same methods as in [2, Lemma 6.2], we can compute the Fréchet-derivatives of E_N and E_R which are given by the formulas

$$\begin{aligned}\langle E_N'(u,v),(h,k)\rangle_{W^*,W} &\\ = \int_\Omega \nabla u \nabla h + \int_\Omega (v + \overline{\lambda(u)})(k + \frac{1}{|\Omega|}(\lambda'(u)|h)_2) + \int_\Omega \Phi'(u)h &\\ + \frac{\sigma_s}{\gamma}\int_\Gamma \nabla_\Gamma u \nabla_\Gamma h + \frac{\kappa}{\gamma}\int_\Gamma uh + \overline{\lambda(u)}(\lambda'(u)|h)_2,&\end{aligned} \tag{5.2}$$

and

$$\langle E_R'(u,v),(h,k)\rangle_{W^*,W} = \int_\Omega \nabla u \nabla h + \int_\Omega vk + \int_\Omega \Phi'(u)h + \frac{\sigma_s}{\gamma}\int_\Omega \nabla_\Gamma u \nabla_\Gamma h + \frac{\kappa}{\gamma}\int_\Gamma uh, \tag{5.3}$$

respectively, for all $(u,v),(h,k) \in W$.

Next we compute the derivative of $E_K(\psi(\cdot),\vartheta(\cdot))$ with respect to time to obtain

$$\frac{d}{dt}\ E_K(\psi(t),\vartheta(t)) = -|\nabla\mu|_2^2 - |\nabla\vartheta|_2^2 - \frac{1}{\gamma}|\partial_t\psi|_{2,\Gamma}^2 - \alpha|\vartheta|_{2,\Gamma}^2. \tag{5.4}$$

Note that in case of Robin boundary conditions, this is an easy consequence of (4.4), whereas in case of Neumann boundary conditions this follows from the fact that $\int_\Omega(\vartheta + \overline{\lambda(\psi)}) = 0$. Furthermore we observe that by (4.6) the functionals E_K are bounded from below. Then (5.4) and the boundedness of E_K yield

$$\psi \in C_b(\mathbb{R}_+; H_2^1(\Omega)) \cap C_b(\mathbb{R}_+; H_2^1(\Gamma)) \quad \text{and} \quad \vartheta \in C_b(\mathbb{R}_+; L_2(\Omega)). \tag{5.5}$$

Proposition 5.1. *Let $(\psi,\vartheta) \in (Z^1 \cap Z_\Gamma^1) \times Z^2$ be a global solution of* (5.1). *Then $(\psi(t),\vartheta(t))$ has relatively compact range in $(H_2^1(\Omega) \cap H_2^1(\Gamma)) \times L_2(\Omega)$.*

Proof. We already know that a global solution is bounded in $(H_2^1(\Omega) \cap H_2^1(\Gamma)) \times L_2(\Omega)$. To prove the relative compactness of the orbit $\psi(\mathbb{R}_+)$ we will proceed in two steps. First we consider the operator $A_p := \Delta^2$ in $L_p(\Omega)$, with domain

$$D(A_p) = \{w \in H_p^4(\Omega) : \Delta w = 0 \text{ and } \partial_\nu w = 0 \text{ on } \Gamma\}.$$

By [2, Proof of Proposition 5.2 (b)] we have $L_p(\Omega) = N(A_p) \oplus R(A_p)$ and the semigroup, generated by A_p is exponentially stable on $R(A_p)$. Let P be the corresponding projection onto $N(A_p)$ and set $Q = I - P$. Consider the evolution equation

$$\partial_t \psi_1 + A_p \psi_1 = Q(\Phi'(\psi) - \lambda'(\psi)\vartheta),\ t > 0, \quad \psi_1(0) = \psi_{10},$$

where ψ_{10} denotes the solution of the elliptic problem

$$\begin{cases} \Delta\psi_{10} = \psi_0,\ x \in \Omega, \\ \partial_\nu \psi_{10} = 0,\ x \in \Gamma. \end{cases}, \quad \int_\Omega \psi_{10} = 0.$$

Since ψ_0 has mean 0 the compatibility condition is fulfilled. Choosing $p = 2$ if $\beta \in (0,1]$ and $p = 6/(\beta+2)$ if $\beta \geq 1$ and using the fact that $\lambda' \in L_\infty(\mathbb{R})$, we obtain $Q(\Phi'(\psi) + \lambda'(\psi)\vartheta) \in C_b(\mathbb{R}_+; R(A_p))$. Thus by semigroup-theory this yields $\psi_1 \in C_b(\mathbb{R}_+; H_p^r(\Omega))$, for each $r < 4$ and by compact embedding the orbit $\Delta\psi_1(\mathbb{R}_+)$ is relatively compact in $H_2^1(\Omega)$.

Next we split ψ by means of $\psi = \Delta\psi_1 + \psi_2$. From [2, Proof of Proposition 5.2] it follows immediately that the orbit $\psi_2(\mathbb{R}_+)$ is relatively compact in $H_2^1(\Omega) \cap H_2^1(\Gamma)$. Therefore the orbit of ψ is relatively compact in $H_2^1(\Omega) \cap H_2^1(\Gamma)$, too.

Now let $e = \vartheta + \lambda(\psi)$. Then e solves the following system

$$\begin{aligned} \partial_t e - \Delta e &= -\Delta\lambda(\psi), & t > 0,\ x \in \Omega, \\ \alpha e + \partial_\nu e &= \alpha\lambda(\psi) + \partial_\nu \lambda(\psi), & t > 0,\ x \in \Gamma, \\ e(0) &= e_0, & t = 0,\ x \in \Omega, \end{aligned}$$

where $e_0 := \vartheta_0 + \lambda(\psi_0)$. By (5.5) we see that $-\Delta\lambda(\psi) \in C_b(\mathbb{R}_+; H_2^1(\Omega)^*)$. The Laplacian generates an exponentially stable analytic semigroup in $H_2^1(\Omega)^*$, if $\alpha > 0$. In case $\alpha = 0$ the semigroup is exponentially stable on $\hat{H}_2^1(\Omega)^*$, where

$$\hat{H}_2^1(\Omega) := \{w \in H_2^1(\Omega) : \int_\Omega w = 0\}.$$

Therefore semigroup-theory implies that $e \in C_b(\mathbb{R}_+; H_2^r(\Omega))$, for every $r \in (0,1)$. One more time we use (5.5) to obtain

$$\vartheta = e - \lambda(\psi) \in C_b(\mathbb{R}_+; H_2^r(\Omega)),$$

hence by compact embedding the orbit $\vartheta(\mathbb{R}_+)$ is relatively compact in $L_2(\Omega)$. □

We remark that the corresponding stationary system to (5.1), with $h = 0$ is given by

$$\begin{cases} -\Delta u + \Phi'(u) - \lambda'(u)c^* = \text{const}, & x \in \Omega, \\ -\sigma_s \Delta_\Gamma u + \gamma\partial_\nu u + \kappa u = 0, & x \in \Gamma, \end{cases} \tag{5.6}$$

where $c^* = \text{const}$, if $\alpha = 0$ and $c^* = 0$, if $\alpha > 0$. Before we consider the next proposition, which justifies forthcoming calculations, we need some preliminaries.

First we rewrite E_N in the following way

$$E_N(u,v) = \frac{1}{2}a((u,v),(u,v)) + \int_\Omega \Phi(u) + \overline{\lambda(u)}\int_\Omega v + |\Omega|(\overline{\lambda(u)})^2, \tag{5.7}$$

with

$$a((u_1,v_1),(u_2,v_2)) = \int_\Omega \nabla u_1 \nabla u_2 + \frac{\sigma_s}{\gamma}\int_\Gamma \nabla_\Gamma u_1 \nabla_\Gamma u_2 + \frac{\kappa}{\gamma}\int_\Gamma u_1 u_2 + \int_\Omega v_1 v_2.$$

The form a is bounded, symmetric, positive and elliptic. The latter property follows from the trace map $H_2^1(\Omega)$ to $L_2(\Gamma)$. By the Lemma of Lax-Milgram there exists a bounded operator $A : W \to W^*$, with $\rho(A) \neq \varnothing$ and we have

$$a((u_1,v_1),(u_2,v_2)) = \langle A(u_1,v_1),(u_2,v_2)\rangle_{W^*,W}.$$

Furthermore we denote by A_p the part of the operator A in the space

$$V_p = \{(u,v) \in L_p(\Omega) \times L_p(\Omega) : \int_\Omega u = 0\}$$

and we set $H = V_2$.

Proposition 5.2. *Let $K \in \{N,R\}$. For every $(u,v) \in W$ the following assertions are equivalent:*

(i) *(u,v) is a critical point of E_K, i.e., $E_K'(u,v) = 0$, with the additional constraint $\int_\Omega (v + \lambda(u)) = 0$ in case $K = N$.*
(ii) *$(u,v) \in D(A_2)$ and (u,v) satisfy (5.6), where $v = c^* = \mathrm{const}$, $c^* = 0$ if $\alpha > 0$.*

Proof. The proof for E_R is nearly the same as in [2, Proposition 6.4]. Therefore we will only consider the case $K = N$. Let (u,v) be a critical point of E_N with the constraint $\int_\Omega (v + \lambda(u)) = 0$. By [2, Lemma 6.3] it holds that

$$D(A_p) = \{(u,v) \in V_p : u \in H_p^2(\Omega) \cap W_p^{3-1/p}(\Gamma),\ -\sigma_s \Delta_\Gamma u + \gamma \partial_\nu u + \kappa u = 0\}.$$

Next by (5.7) we obtain

$$A(u,v) = \left((h,k) \mapsto -\int_\Omega \left(\Phi'(u) + \lambda'(u)(2\overline{\lambda(u)} + \bar{v})\right) h - \int_\Omega \overline{\lambda(u)} k\right),$$

hence $A(u,v) \in V_p$, where p can be chosen strictly larger than $6/5$, due to (4.3). Then a bootstrap argument and the characterization of $D(A_p)$ lead to $A(u,v) \in H$. Finally we use integration by parts to obtain

$$\begin{aligned}\langle E_N'(u,v),(h,k)\rangle_{W^*,W} &= \int_\Omega (-\Delta u + \Phi'(u) - \lambda'(u)v)h \\ &+ \frac{1}{\gamma}\int_\Gamma (-\sigma_s \Delta_\Gamma + \gamma \partial_\nu u + \kappa u)h + \int_\Omega (v + \overline{\lambda(u)})(k + \lambda'(u)h),\end{aligned} \tag{5.8}$$

for all $(h,k) \in W$. This yields (ii). The implication (ii)$\Rightarrow$ (i) is a simple consequence of (5.8). □

Remark. In other words, the statement of Proposition 5.2 means that every weak solution of the stationary system (5.6) is already a strong solution and vice versa.

Assuming in addition that Φ is real analytic and that in case of Neumann boundary conditions λ is real analytic too, we obtain the following result.

Proposition 5.3 (Lojasiewicz-Simon inequality). *Let $K \in \{N, R\}$ and let $(\varphi, \theta) \in W$ be a critical point of the functional E_K. Assume that Φ is real analytic and in case $K = N$, λ is real analytic too, and let* (4.3) *as well as $\lambda', \lambda'', \lambda''' \in L_\infty(\mathbb{R})$ hold. Then there exist constants $s \in (0, \frac{1}{2}], C, \delta > 0$ such that*

$$|E_K(u, v) - E_K(\varphi, \theta)|^{1-s} \leq C|E'_K(u, v)|_{W^*},$$

whenever $|(u, v) - (\varphi, \psi)|_W \leq \delta$.

Proof. The arguments used in [2, Proposition 6.6] for a proof of this assertion carry over to our case, since $\lambda', \lambda'', \lambda''' \in L_\infty(\mathbb{R})$. We will not repeat them here. □

The next proposition collects some properties of the functionals $E_K : W \to \mathbb{R}$.

Proposition 5.4. *Let (ψ, ϑ) be a global solution of* (5.1) *with $h = \frac{1}{|\Omega|}\int \psi_0$ and suppose that Φ satisfies* (4.1) *as well as* (4.3)*. Let further $K \in \{N, R\}$. Then the following statements hold.*

(i) *The functions $E_K(\psi(\cdot), \vartheta(\cdot))$ are nonincreasing and the limits*

$$\lim_{t\to\infty} E_K(\psi(t), \vartheta(t)) =: E_K^\infty$$

exist.

(ii) *The ω-limit set*

$$\omega(\psi, \vartheta) := \{(\varphi, \theta) \in W : \exists\ (t_n)_{n\in\mathbb{N}} \nearrow \infty,\ s.t.\ (\psi(t_n), \vartheta(t_n)) \to (\varphi, \theta) \ in\ W\}$$

is nonempty, compact, connected and E_K is constant on $\omega(\psi, \vartheta)$.

(iii) *For every $(\varphi, \theta) \in \omega(\psi, \vartheta)$ it holds that $\theta = c^*$ and (φ, θ) is a solution of the stationary problem* (5.6)*.*

(iv) *Every $(\varphi, \theta) \in \omega(\psi, \vartheta)$ is a critical point of E_K, i.e., $E'_K(\varphi, \theta) = 0$.*

Proof. By (5.4) the functions $E_N(\psi(\cdot), \vartheta(\cdot))$ and $E_R(\psi(\cdot), \vartheta(\cdot))$ are nonincreasing, hence the limits $\lim_{t\to\infty} E_K(\psi(t), \vartheta(t))$ exist, since $E_K(\psi(\cdot), \vartheta(\cdot))$ are bounded from below. This yields (i). By Proposition 5.1, the solution (ψ, ϑ) has relatively compact range in W. Therefore, by well-known results, the ω-limit set is nonempty, compact and connected. The fact that E_K is constant on the ω-limit set, follows easily from continuity of E_K on W and (i).

Now let $(\varphi, \theta) \in \omega(\psi, \vartheta)$ and let $(t_n)_{n\in\mathbb{N}} \nearrow \infty$, such that $(\psi(t_n), \vartheta(t_n)) \to (\varphi, \theta)$ in W, as $n \to \infty$. From (5.4) and the boundedness of $E_K(\psi(\cdot), \vartheta(\cdot))$ we obtain

$$\nabla\mu \in L_2(\mathbb{R}_+; L_2(\Omega)),\ \partial_t\psi|_\Gamma \in L_2(\mathbb{R}_+; L_2(\Gamma)) \quad \text{and} \quad \nabla\vartheta \in L_2(\mathbb{R}_+; L_2(\Omega)).$$

This implies $\partial_t\psi, \partial_t\vartheta \in L_2(\mathbb{R}_+; H_2^1(\Omega)^*)$ and for every $s \in [0,1]$,

$$\lim_{n\to\infty} \psi(t_n+s) = \varphi \text{ in } H_2^1(\Omega)^*,$$

$$\lim_{n\to\infty} \psi(t_n+s) = \varphi \text{ in } L_2(\Gamma) \quad \text{and} \quad \lim_{n\to\infty} \vartheta(t_n+s) = \theta \text{ in } H_2^1(\Omega)^*.$$

Since (ψ,ϑ) has relatively compact range in W, the above three limits exist even in $H_2^1(\Omega)$, $H_2^1(\Gamma)$ and $L_2(\Omega)$, respectively. Integrating the equations $(5.1)_1$–$(5.1)_5$ we finally obtain that $\theta = c^* =$ const and (φ, c^*) solves (5.6), hence the assertion (iii).

Now we turn to (iv). Suppose that $(u,v) = (\varphi,\theta) \in \omega(\psi,\vartheta)$. Then by (iii), $\theta = c^*$ and (φ, c^*) solves (5.6). Proposition 5.2, (5.3) and integration by parts lead to

$$\langle E_N'(\varphi,c^*),(h,k)\rangle_{W^*,W} = (c^* + \overline{\lambda(\varphi)}) \int_\Omega (k + \lambda'(\varphi)h) = 0,$$

and

$$\langle E_R'(\varphi,c^*),(h,k)\rangle_{W^*,W} = c^* \int_\Omega (k + \lambda'(\varphi)h) = 0,$$

since in case of Robin boundary conditions we have $c^* = 0$ and in the other case we have $\int_\Omega(\vartheta + \overline{\lambda(\psi)}) = \int_\Omega(\vartheta + \lambda(\psi)) = 0$ and for some sequence $t_n \to \infty$ it holds $(\psi(t_n), \vartheta(t_n)) \to (\varphi, c^*)$ as $n \to \infty$ in W, hence $\int_\Omega(c^* + \overline{\lambda(\varphi)}) = 0$, and therefore $c^* + \overline{\lambda(\varphi)} = 0$. The proposition is proved. □

Now we are in a position to state our main result concerning the asymptotic behavior of solutions of the Cahn-Hilliard equation.

Theorem 5.5. *Let (ψ,ϑ) be a global solution of the Cahn-Hilliard equation* (5.1) *with $h = \frac{1}{|\Omega|}\int_\Omega \psi_0$ and suppose that Φ satisfies the conditions* (4.1) *as well as* (4.3) *and let $\lambda', \lambda'', \lambda''' \in L_\infty(\mathbb{R})$. Assume that Φ is real analytic, and that λ is real analytic, if $\alpha = 0$. Then the limits*

$$\lim_{t\to\infty} \psi(t) =: \varphi, \quad \textit{and} \quad \lim_{t\to\infty} \vartheta(t) =: c^* = \text{const}$$

exist in $H_2^1(\Omega) \cap H_2^1(\Gamma)$ and $L_2(\Omega)$, respectively, and (φ, c^*) is a solution of the stationary problem (5.6).

Proof. Proposition 5.3 yields, that for every $(\varphi,\theta) \in \omega(\psi,\vartheta)$ there exist constants $s \in (0, \frac{1}{2}]$, $C > 0$ and $\delta > 0$ such that

$$|E_K(u,v) - E_K(\varphi,\theta)|^{1-s} \le C|E_K'(u,v)|_{W^*},$$

whenever $|(u,v) - (\varphi,\theta)|_W \le \delta$. By Proposition 5.4 (iii) the ω-limit set $\omega(\psi,\vartheta)$ is compact, hence we may cover it by a union of *finitely* many balls with center (φ_i,θ_i) and radius δ_i, $i = 1,\ldots,N$. Since $E_K(u,v) \equiv E_K^\infty$ on $\omega(\psi,\vartheta)$, there are *uniform* constants $s \in (0,\frac{1}{2}]$, $C > 0$ and an open set $U \supset \omega(\psi,\vartheta)$, with

$$|E_K(u,v) - E_K^\infty|^{1-s} \le C|E_K'(u,v)|_{W^*}, \tag{5.9}$$

for all $(u,v) \in U$. After these preliminaries, we define the function $H: \mathbb{R}_+ \to \mathbb{R}_+$ by

$$H(t) := (E_K(\psi(t),\vartheta(t)) - E_K^\infty)^s.$$

By Proposition 5.4 the function H is nonincreasing and $\lim_{t\to\infty} H(t) = 0$. A well-known result in the theory of dynamical systems implies further that

$$\lim_{t\to\infty} \operatorname{dist}((\psi(t), \vartheta(t)), \omega(\psi, \vartheta)) = 0,$$

i.e., there exists $t^* \geq 0$, such that $(\psi(t), \vartheta(t)) \in U$, whenever $t \geq t^*$. Next, we compute and estimate the time derivative of H. By (5.4) and (5.9) we obtain

$$\begin{aligned} -\frac{d}{dt} H(t) &= s\left(-\frac{d}{dt} E_K(\psi(t), \vartheta(t))\right) |E_K(\psi(t), \vartheta(t)) - E_K^\infty|^{s-1} \\ &\geq C \frac{|\nabla\mu|_2^2 + |\nabla\vartheta|_2^2 + |\partial_t\psi|_{2,\Gamma}^2 + \alpha|\vartheta|_{2,\Gamma}^2}{|E_K'(\psi(t), \vartheta(t))|_{W^*}}. \end{aligned} \tag{5.10}$$

Together with the Cauchy-Schwarz inequality, integration by parts, as well as by (5.2) and by (5.3), we obtain

$$\begin{aligned} &|\langle E_N'(\psi, \vartheta), (h, k)\rangle_{W^*, W}| \\ &\quad = |\int_\Omega (\mu - \overline{\mu})h + \int_\Omega (\vartheta + \overline{\lambda(\psi)})(k + \lambda'(\psi)h) - \frac{1}{\gamma}\int_\Gamma \partial_t\psi h| \\ &\quad \leq C(|\nabla\mu|_2|h|_2 + (|h|_2 + |k|_2)|\nabla\vartheta|_2 + |\partial_t\psi|_{2,\Gamma}|h|_{H_2^1(\Omega)}), \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} &|\langle E_R'(\psi, \vartheta), (h, k)\rangle_{W^*, W}| \\ &\quad = |\int_\Omega (\mu - \overline{\mu})h + \int_\Omega \vartheta(k + \lambda'(\psi)h) - \frac{1}{\gamma}\int_\Gamma \partial_t\psi h| \\ &\quad \leq C(|\nabla\mu|_2|h|_2 + (|h|_2 + |k|_2)(|\nabla\vartheta|_2 + \alpha^{1/2}|\vartheta|_{2,\Gamma}) + |\partial_t\psi|_{2,\Gamma}|h|_{H_2^1(\Omega)}), \end{aligned} \tag{5.12}$$

respectively, where $\bar{\mu} = \frac{1}{|\Omega|}\int_\Omega \mu$. Here we applied the Poincaré inequality to the functions $\mu - \bar{\mu}$ and $\vartheta + \overline{\lambda(\psi)}$, since their means are 0. Concerning ϑ in E_R', we made use of a version of the Poincaré inequality, namely

$$|w|_2 \leq C(|\nabla w|_2 + |w|_{2,\Gamma}), \quad w \in H_2^1(\Omega).$$

If we take the supremum in (5.11) and (5.12) over all $(h, k) \in W$ with norm less than 1 this results in

$$|E_K'(\psi(t), \vartheta(t))|_{W^*} \leq C(|\nabla\mu(t)|_2 + |\nabla\vartheta(t)|_2 + \alpha^{1/2}|\vartheta(t)|_{2,\Gamma} + |\partial_t\psi(t)|_{2,\Gamma}).$$

Hence from (5.10) it follows that

$$-\frac{d}{dt} H(t) \geq C(|\nabla\mu(t)|_2 + |\nabla\vartheta(t)|_2 + \alpha^{1/2}|\vartheta(t)|_{2,\Gamma} + |\partial_t\psi(t)|_{2,\Gamma})$$

and this in turn implies that

$$\nabla\mu, \nabla\vartheta \in L_1([t^*, \infty), L_2(\Omega)) \quad \text{and} \quad \vartheta, \partial_t\psi \in L_1([t^*, \infty); L_2(\Gamma)).$$

Furthermore this yields $\partial_t\psi, \partial_t\vartheta \in L_1([t^*, \infty), H_2^1(\Omega)^*)$, hence the limits for $t \to \infty$ exist in $L_2(\Gamma)$ and $H_2^1(\Omega)^*$, respectively. Therefore the claim follows from relative compactness of $(\psi(t), \vartheta(t))$ in $(H_2^1(\Omega) \cap H_2^1(\Gamma)) \times L_2(\Omega)$ and Proposition 5.4 (iii).

$\square$

6. Appendix

(a) *Proof of Proposition* 3.2.

(i) By Hölders inequality it holds that

$$\begin{aligned}
&|\Delta\Phi'(u) - \Delta\Phi'(v)|_p \\
&\quad \le |\Delta u\Phi''(u) - \Delta v\Phi''(v)|_p + ||\nabla u|^2\Phi'''(u) - |\nabla v|^2\Phi'''(v)|_p \\
&\quad \le |\Delta u|_{rp}|\Phi''(u) - \Phi''(v)|_{r'p} + |\Delta u - \Delta v|_{rp}|\Phi''(v)|_{r'p} \\
&\qquad + |\nabla u|_{2\sigma p}|\Phi'''(u) - \Phi'''(v)|_{\sigma' p} + ||\nabla u|^2 - |\nabla v|^2|_{\sigma p}|\Phi'''(v)|_{\sigma' p} \\
&\quad \le T^{1/r'p}\left(|\Delta u|_{rp}|\Phi''(u) - \Phi''(v)|_\infty + |\Delta u - \Delta v|_{rp}|\Phi''(v)|_\infty\right) \\
&\qquad + T^{1/\sigma' p}\left(|\nabla u|_{2\sigma p}|\Phi'''(u) - \Phi'''(v)|_\infty + ||\nabla u|^2 - |\nabla v|^2|_{\sigma p}|\Phi'''(v)|_\infty\right),
\end{aligned}$$

where we also used the fact that all functions belonging to $\mathbb{B}_R(u^*)$ are uniformly bounded. We have

$$\nabla w \in H_p^{3\theta_1/4}(J; H_p^{3(1-\theta_1)}(\Omega)) \hookrightarrow L_{2\sigma p}(J \times \Omega), \quad \theta_1 \in [0,1],$$

and

$$\Delta w \in H_p^{\theta_2/2}(J; H_p^{2(1-\theta_2)}(\Omega)) \hookrightarrow L_{rp}(J \times \Omega), \quad \theta_2 \in [0,1],$$

for every function $w \in \mathbb{B}_R(u^*)$, since $r, \sigma > 1$ may be chosen close to 1. Therefore we have

$$|\Delta\Phi'(u) - \Delta\Phi'(v)|_p \le \mu_1(T)\,(R + |u^*|_1)\,|u - v|_1,$$

due to the assumption $\Phi \in C^{4-}(\mathbb{R})$. The function μ_1 is given by

$$\mu_1(T) = \max\{T^{1/r'p},\ T^{1/\sigma' p}\}.$$

This yields (i).

(ii) Firstly we observe that

$$\begin{aligned}
&\Delta(\lambda'(w)F(w)) \\
&\quad = (\Delta w\lambda''(w) + |\nabla w|^2\lambda'''(w))F(w) + 2\lambda''(w)\nabla w \cdot \nabla F(w) + \lambda'(w)\Delta F(w),
\end{aligned}$$

for all $w \in \mathbb{B}_R(u^*)$. Secondly by (3.6), the embeddings

$$F(w) \in H_p^{s+\theta-1}(H_p^{3-2\theta}) \hookrightarrow L_{2p}(J \times \Omega)$$

and

$$\nabla F(w) \in H_p^{s+\theta-1}(H_p^{2-2\theta}) \hookrightarrow L_{4p/3}(J \times \Omega),$$

with $s \in \left[\frac{1}{2}, \frac{3}{4}\right)$, $\theta \in [0,1]$, are valid, whenever

$$p > \frac{2}{2s+1}\left(\frac{n}{4} + \frac{1}{2}\right) \quad \text{and} \quad p > \frac{1}{s}\left(\frac{n}{8} + \frac{1}{4}\right),$$

respectively. It is obvious, that these conditions are fulfilled for every $s \in \left[\frac{1}{2}, \frac{3}{4}\right)$, whenever $p > n/4 + 1$. An easy computation shows that $\nabla w \in L_{4p}(J \times \Omega)$ and $\Delta w \in L_{2p}(J \times \Omega)$, if $p > n/4 + 1$ (here we use strict embeddings). If $1/\sigma + 1/\sigma' = 1$

and $\sigma > 1$ is sufficiently small, then Hölder's inequality and Proposition 3.1 lead to the estimate

$$\begin{aligned}
&|\lambda''(u)\Delta u F(u) - \lambda''(v)\Delta v F(v)|_p \\
&\quad \le C|\Delta u F(u) - \Delta v F(v)|_p + |\lambda''(u) - \lambda''(v)|_\infty |\Delta v F(v)|_p \\
&\quad \le T^{1/2\sigma' p}(|\Delta u|_{2\sigma p}|F(u) - F(v)|_{2p} + |\Delta u - \Delta v|_{2\sigma p}|F(v)|_{2p} \\
&\qquad\qquad + |\lambda''(u) - \lambda''(v)|_\infty |\Delta v|_{2\sigma p}|F(v)|_{2p}) \\
&\quad \le \mu_2(T)(1 + |u^*|_1)|u - v|_1.
\end{aligned}$$

In a similar way we obtain

$$|\lambda'''(u)|\nabla u|^2 F(u) - \lambda'''(v)|\nabla v|^2 F(v)|_p \le \mu_2(T)(1 + |u^*|_1)|u - v|_1,$$

$$|\lambda''(u)\nabla u \nabla F(u) - \lambda''(v)\nabla v \nabla F(v)|_p \le \mu_2(T)(1 + |u^*|_1)|u - v|_1,$$

as well as

$$|\lambda'(u)\Delta F(u) - \lambda'(v)\Delta F(v)|_p \le \mu_2(T)(1 + |u^*|_1)|u - v|_1,$$

for all $u, v \in \mathbb{B}_R(u^*)$. This proves (ii).

(iii) This is an easy consequence of (i) and (3.5), since by trace-theory (cf. [4]) we obtain

$$|\partial_\nu \Phi'(u) - \partial_\nu \Phi'(v)|_{Y_1(T)} \le C\left(|\Phi'(u) - \Phi'(v)|_{H_p^{1/2}(L_p)} + |\Phi'(u) - \Phi'(v)|_{L_p(H_p^2)}\right).$$

(iv) In a similar way as in (iii) we obtain

$$\begin{aligned}
|\partial_\nu(\lambda'(u)F(u)) - \partial_\nu(\lambda'(v)F(v))|_{Y_1(T)} \le C\Big(&|\lambda'(u)F(u) - \lambda'(v)F(v)|_{H_p^{1/2}(L_p)} \\
&+|\lambda'(u)F(u) - \lambda'(v)F(v)|_{L_p(H_p^2)}\Big).
\end{aligned}$$

The desired Lipschitz-estimate for the second term follows from (ii). The first term will be rewritten in the usual way, i.e.,

$$\begin{aligned}
&|\lambda'(u)F(u) - \lambda'(v)F(v)|_{H_p^{1/2}(L_p)} \\
&\qquad \le \left(|(\lambda'(u) - \lambda'(v))F(u)|_{H_p^{1/2}(L_p)} + |\lambda'(v)(F(u) - F(v))|_{H_p^{1/2}(L_p)}\right).
\end{aligned}$$

Applying (3.4) we obtain

$$\begin{aligned}
|(\lambda'(u) - \lambda'(v))F(u)|_{H_p^{1/2}(L_p)} \le C\Big(&|\lambda'(u) - \lambda'(v)|_{H_{rp}^{1/2}(L_{rp})}|F(u)|_{L_{r'p}(L_{r'p})} \\
&+T^{1/\sigma' p}|\lambda'(u) - \lambda'(v)|_\infty |F(u)|_{H_{\sigma p}^{1/2}(L_{\sigma p})}\Big),
\end{aligned}$$

as well as

$$\begin{aligned}
|\lambda'(v)(F(u) - F(v))|_{H_p^{1/2}(L_p)} \le C\Big(&|\lambda'(v)|_{H_{rp}^{1/2}(L_{rp})}|F(u) - F(v)|_{L_{r'p}(L_{r'p})} \\
&+T^{1/\sigma' p}|\lambda'(v)|_\infty |F(u) - F(v)|_{H_{\sigma p}^{1/2}(L_{\sigma p})}\Big).
\end{aligned}$$

Let $w \in \mathbb{B}_R(u^*)$. It is obvious that $F(w) \in H^{1/2}_{\sigma p}(J; L_{\sigma p}(\Omega))$, since $\sigma > 1$ may be chosen arbitrarily close to 1. So it remains to check if $\lambda'(w) \in H^{1/2}_{rp}(J; L_{rp}(\Omega))$ and $F(w) \in L_{r'p}(J; L_{r'p}(\Omega))$. It holds

$$H^{\theta}_p(J; H^{4(1-\theta)}_p(\Omega)) \hookrightarrow H^{1/2}_{rp}(J; L_{rp}(\Omega))$$

and

$$H^{s+\theta-1}_p(J; H^{3-2\theta}_p(\Omega)) \hookrightarrow L_{r'p}(J; L_{r'p}(\Omega))$$

if $p > \frac{2}{r'}(\frac{n}{4}+1)$ and $p > \frac{2}{r(2s+1)}(\frac{n}{2}+1)$, respectively. Thus we set $r' = r = 2$. Now the claim follows from (3.5) and Proposition 3.1.

(v) With the help of Hölder's inequality we compute

$$\begin{aligned} &|(\Delta\lambda'(u) - \Delta\lambda'(v))\eta|_p \qquad (6.1)\\ &\qquad \le |\eta|_{Z^2}(|\Delta u\lambda''(u) - \Delta v\lambda''(v)|_{2p} + ||\nabla u|^2\lambda'''(u) - |\nabla v|^2\lambda'''(v)|_{2p}), \end{aligned}$$

for each $\eta \in Z^2$. Since $\lambda \in C^{4-}(\mathbb{R})$, it follows from the uniform boundedness of $u, v \in \mathbb{B}_R(u^*)$ that

$$\begin{aligned} |\Delta u\lambda''(u) - \Delta v\lambda''(v)|_{2p} &\le |\lambda''(u) - \lambda''(v)|_\infty |\Delta u|_{2p} + |\lambda''(v)|_\infty |\Delta u - \Delta v|_{2p}\\ &\le C(1 + |u^*|_1)|u - v|_1. \end{aligned}$$

In a similar way the second term in (6.1) can be treated, obtaining

$$||\nabla u|^2\lambda'''(u) - |\nabla v|^2\lambda'''(v)|_{2p} \le C(1 + |u^*|_1)|u - v|_1.$$

Furthermore we have

$$\begin{aligned} |(\nabla u\lambda''(u) - \nabla v\lambda''(v))\nabla\eta|_p &\le |\nabla\eta|_{4p/3}|\nabla u\lambda''(u) - \nabla v\lambda''(v)|_{4p}\\ &\le |\eta|_{Z^2}(|\lambda''(u) - \lambda''(v)|_\infty |\nabla u|_{4p} + |\lambda''(v)|_\infty |\nabla u - \nabla v|_{4p})\\ &\le C|\eta|_{Z^2}(1 + |u^*|_1)|u - v|_1 \end{aligned}$$

and

$$|(\lambda'(u) - \lambda'(v))\Delta\eta|_p \le |\lambda'(u) - \lambda'(v)|_\infty |\Delta\eta|_p \le C|\eta|_{Z^2}|u - v|_1,$$

by Hölder's inequality and the Lipschitz-property of λ', λ''.

(vi) Finally we apply trace-theory to obtain

$$\begin{aligned} &|\partial_\nu(\lambda'(u)\eta) - \partial_\nu(\lambda'(v)\eta)|_{Y_1(T)}\\ &\qquad \le C\left(|(\lambda'(u) - \lambda'(v))\eta|_{H^{1/2}_p(L_p)} + |(\lambda'(u) - \lambda'(v))\eta|_{L_p(H^2_p)}\right). \end{aligned}$$

The estimate for the second term is clear by (v). Again we will use (3.4) to estimate the first term. This yields

$$\begin{aligned} |(\lambda'(u) - \lambda'(v))\eta|_{H^{1/2}_p(L_p)} \le C(T_0)\Big(&|\lambda'(u) - \lambda'(v)|_{H^{1/2}_{rp}(L_{rp})}|\eta|_{L_{r'p}(L_{r'p})}\\ &+|\lambda'(u) - \lambda'(v)|_{L_{\sigma'p}(L_{\sigma'p})}|\eta|_{H^{1/2}_{\sigma p}(L_{\sigma p})}\Big). \end{aligned}$$

As in (iv), it follows that $r = 2$. Furthermore we have

$$H^{\theta}_p(J; H^{2(1-\theta)}_p(\Omega)) \hookrightarrow L_{2p}(J; L_{2p}(\Omega)),$$

if $p > \frac{n}{4} + 1$. Last but not least we apply (3.5). The proof is complete.

(b) *Proof of Lemma* 4.1.

Step 1. We start with $\Delta\Phi'(\psi) = \Delta\psi\Phi''(\psi) + |\nabla\psi|^2\Phi'''(\psi)$. Using the Gagliardo-Nirenberg inequality and (4.3) we obtain

$$|\Phi''(\psi)\Delta\psi|_p \leq |\psi|^{\beta+1}_{2(\beta+1)p}|\Delta\psi|_{2p} \leq C|\psi|^{a+b(\beta+1)}_{H^4_p}|\psi|_q^{1-a+(1-b)(\beta+1)}, \tag{6.2}$$

where q will be chosen in such a way that $H^1_2(\Omega) \hookrightarrow L_q$, i.e., $\frac{n}{q} \geq \frac{n}{2} - 1$ and

$$(a + (\beta+1)b)\left(4 - \frac{n}{p} + \frac{n}{q}\right) = 2 - \frac{n}{p} + \frac{n}{q}(\beta+2).$$

The second term $\Phi'''(\psi)|\nabla\psi|^2$ will be treated in a similar way. The Gagliardo-Nirenberg inequality and (4.3) again yield

$$|\Phi'''(\psi)|\nabla\psi|^2|_p \leq |\psi|^{\beta}_{2\beta p}|\nabla\psi|^2_{4p} \leq C|\psi|^{2a+b\beta}_{H^4_p}|\psi|_q^{2-2a+(1-b)\beta}, \tag{6.3}$$

with $\frac{n}{q} \geq \frac{n}{2} - 1$ and

$$(2a + \beta b)\left(4 - \frac{n}{p} + \frac{n}{q}\right) = 2 - \frac{n}{p} + (\beta+2)\frac{n}{q}.$$

It turns out that the condition $\beta < 3$ in case $n = 3$ ensures that either $a+(\beta+1)b < 1$ and $2a+\beta b < 1$ in (6.2) and (6.3), respectively. Integrating (6.2) and (6.3) with respect to t and using Hölders inequality we obtain the desired estimate. Now we estimate $\partial_\nu\Phi'(\psi)$ in Y_1. By trace-theory we obtain

$$|\partial_\nu\Phi'(\psi)|_{Y_1} \leq C(|\Phi'(\psi)|_{H^{1/2}_p(L_p)} + |\Phi'(\psi)|_{L_p(H^2_p)}).$$

The estimate in $L_p(H^2_p)$ follows from the considerations above. By the mean-value theorem and (4.3) we obtain

$$|\Phi'(\psi)|_{H^{1/2}_p(L_p)} \leq C(|\psi|^{\beta+2}_{L_{(\beta+2)p}(L_{(\beta+2)p})} + |\psi|_{H^{1/2}_p(L_p)} + |\psi|^{\beta+1}_{L_{\sigma'p}(L_{r'p})}|\psi|_{H^{1/2}_{\sigma p}(L_{rp})}), \tag{6.4}$$

where $1/\sigma + (\beta+1)/\sigma' = 1/r + (\beta+1)/r' = 1$. This follows similarly to (3.4) from the characterization of H^s_p via differences and Hölders inequality. The Gagliardo-Nirenberg inequality implies

$$|\psi|_{L_{\sigma'p}(L_{r'p})} \leq c|\psi|^a_{Z^1}|\psi|^{1-a}_{L_\infty(H^1_2)},$$

if $a \in [0, 1/\sigma']$ and $a(3 - n/p + n/2) \geq n/2 - 1 - n/r'p$. Therefore we set

$$a = 1/\sigma' = [n/2 - 1 - n/r'p]_+/(3 - n/p + n/2)$$

and choose $r' = (\beta+1)n/p$ if $p \leq n$, $r' = 2(\beta+1)n/p$ if $n < p \leq 2n$ and $r' = \infty$ if $p > 2n$. Observe that

$$Z^1 \hookrightarrow H^{1-\theta}_p(H^{4\theta}_p) \hookrightarrow H^s_p(L_{rp}),$$

if $s = 1 - n/4r'p$. Hence complex interpolation yields

$$|\psi|_{H^{1/2}_{\sigma p}(L_{rp})} \leq c|\psi|^b_{Z^1}|\psi|^{1-b}_{L_{\tau p}(L_{rp})},$$

provided $b = 1/2s$ and
$$1/\sigma \geq b + (1-b)/\tau. \tag{6.5}$$
Finally we apply the Gagliardo-Nirenberg inequality one more time to obtain
$$|\psi|_{L_{\tau p}(L_{rp})} \leq c|\psi|^d_{Z^1}|\psi|^{1-d}_{L_\infty(H^1_2)},$$
with $d(3 - n/p + n/2) \geq n/2 - 1 - n/rp$ and $d \in [0, 1/\tau]$. We set
$$d = 1/\tau = [n/2 - 1 - n/rp]_+/(3 - n/p + n/2).$$
Suppose now that (6.5) holds. Then we have
$$(\beta+1)a + b + (1-b)d = \frac{\beta+1}{\sigma'} + b + \frac{1-b}{\tau} \leq \frac{\beta+1}{\sigma'} + \frac{1}{\sigma} = 1.$$
Hence the desired estimate follows if the inequality (6.5) is strict. We have to distinguish three cases, namely $p \leq n$, $n < p \leq 2n$ and $p > 2n$. In the first case we have $r' = (\beta+1)n/p$, $s = (4\beta+3)/(4\beta+4)$ and $b = (2\beta+2)/(4\beta+3)$. Since $p \geq 2$, (6.5) is equivalent to
$$\frac{6\beta+3}{4\beta+3} - (\beta+1)[n/2 - 1/(\beta+1) - 1]_+ \geq 0.$$
We see that $[n/2 - 1/(\beta+1) - 1]_+ = 0$, if either $n = 1, 2$ or $n = 3$ and $\beta \leq 1$; then we are done. So let $n = 3$ and $\beta > 1$. An easy calculation shows that
$$\frac{6\beta+3}{4\beta+3} > \frac{\beta-1}{2},$$
for all $1 < \beta < 3$. In the second case we have $r' = 2(\beta+1)n/p$, $s = (8\beta+7)/(8\beta+8)$ and $b = (4\beta+4)/(8\beta+7)$, thus (6.5) is equivalent to
$$\frac{4\beta+3}{8\beta+7}(3-n/p+n/2) - (\beta+1)[n/2 - 1/2(\beta+1) - 1]_+ \geq \frac{4\beta+3}{8\beta+7}[n/2 - 1/2 - n/p]_+.$$
Since $p \leq 2n$ we see that this inequality is strict for $n = 1, 2$. If $n = 3$ and due to $p > n$, the inequality reduces to
$$\frac{4\beta+3}{8\beta+7} \geq \beta/7$$
and we have again strict inequality, if $\beta < 3$. In the last case, we have $r' = \infty$, $s = 1$ and $b = 1/2$. Therefore (6.5) is equivalent to
$$(3 - n/p + n/2) - 2(\beta+1)[n/2 - 1]_+ \geq [n/2 - n/p - 1]_+.$$
Again for $n = 1, 2$ we have $[n/2 - 1]_+ = [n/2 - n/p - 1]_+ = 0$, thus we set $n = 3$. Since $p > 2n$ this yields
$$4 - (\beta+1) \geq 0,$$
hence strict inequality if $\beta < 3$. Note that the second term on the right-hand side of (6.4) is dominated by the third term. Furthermore the desired estimate for the first term is a simple consequence of the Gagliardo-Nirenberg inequality as long as $q > (\beta+1)n/4$.

Step 2. Next we estimate the term $\Delta(\lambda'(\psi)F(\psi))$ in X. Observe that

$$\Delta(\lambda'(\psi)F(\psi)) = F(\psi)(\Delta\psi\lambda''(\psi)+|\nabla\psi|^2\lambda'''(\psi))+2\lambda''(\psi)\nabla\psi\nabla F(\psi)+\lambda'(\psi)\Delta F(\psi).$$

As in Section 2 we will solve the homogeneous heat equation (2.2). Obviously $|F(\psi)|_{L_p(L_p)} \leq C(1+|\psi|_{L_p(L_p)})$, for every $1<p<\infty$, since $|\lambda(s)| \leq C(1+|s|)$. Applying the Gagliardo-Nirenberg one more time we obtain

$$\begin{aligned}|\lambda''(\psi)F(\psi)\Delta\psi|_{L_p(L_p)} &\leq C(|\Delta\psi|_{L_{3p/2}(L_{3p/2})} + |\Delta\psi|_{L_{3p/2}(L_{3p/2})}|\psi|_{L_{3p}(L_{3p})}) \\ &\leq C(|\psi|^a_{Z^1}|\psi|^{1-a}_{L_\infty(L_q)} + |\psi|^{a+b}_{Z^1}|\psi|^{2-(a+b)}_{L_\infty(L_q)}),\end{aligned}$$

if $a(4-n/p+n/q) = 2-2n/3p+n/q$ and $b(4-n/p+n/q) = n/q-n/3p$ and $a\in[1/2,1]$, $b\in[0,1]$. The two latter conditions are fulfilled if $q\leq 3p$. We require furthermore $a<2/3$ and $b<1/3$. This leads to the condition $q>n/2$, which is true. Then we also have $a+b<1$. In a similar way we estimate $\lambda'''(\psi)F(\psi)|\nabla\psi|^2$, to obtain

$$\begin{aligned}|\lambda'''(\psi)F(\psi)|\nabla\psi|^2|_{L_p(L_p)} &\leq C(1+|\psi|_{L_{3p}(L_{3p})})|\nabla\psi|^2_{L_{3p}(L_{3p})} \\ &\leq C(|\psi|^{2a}_{Z^1}|\psi|^{2(1-a)}_{L_\infty(L_q)} + |\psi|^{2a+b}_{Z^1}|\psi|^{3-(2a+b)}_{L_\infty(L_q)}),\end{aligned} \tag{6.6}$$

whenever $a(4-n/p+n/q) = 1+n/q-n/3p$ and $b(4-n/p+n/q) = n/q-n/3p$ and $a\in[1/4,1]$, $b\in[0,1]$. The two latter conditions are satisfied if $q\leq 3p$. It is easy to verify that $a\leq 1/3$ and $b<1/3$, whenever $q\geq 2n$, i.e., $q=6$. Finally it holds $2a+b<1$.

Note that the representation of $F(\psi)$ implies

$$|\nabla F(\psi)|_{L_{2p}(L_{2p})} \leq C(1+|\nabla\lambda(\psi)|_{L_{2p}(L_{2p})}) \leq C(1+|\nabla\psi|_{L_{2p}(L_{2p})}),$$

hence by the inequalities of Hölder and Gagliardo-Nirenberg we obtain

$$\begin{aligned}|\lambda''(\psi)\nabla F(\psi)\nabla\psi|_{L_p(L_p)} &\leq |\nabla F(\psi)|_{L_{2p}(L_{2p})}|\nabla\psi|_{L_{2p}(L_{2p})} \\ &\leq C(1+|\nabla\psi|^2_{L_{2p}(L_{2p})}) \leq C(1+|\psi|^{2a}_{Z^1}|\psi|^{2(1-a)}_{L_\infty(L_q)}),\end{aligned}$$

with

$$2a\left(4-\frac{n}{p}+\frac{n}{q}\right) = 2+\frac{2n}{q}-\frac{n}{p}, \quad a\in[1/4,1].$$

Since $q\leq 3p$ we see that $a\geq 1/4$. Furthermore we have $2a<1$, if $n<6$. The estimate of $\lambda'(\psi)\Delta F(\psi)$ in L_p is more involved. With the help of (3.6) with $s=\theta=1/2$, we obtain

$$\begin{aligned}|\lambda'(\psi)\Delta F(\psi)|_{L_p(L_p)} &\leq C(1+|\lambda(\psi)|_{H^{1/2}_p(H^1_p)}) \\ &\leq C(1+|\psi|_{H^{1/2}_p(L_p)} + |\nabla\psi\lambda'(\psi)|_{H^{1/2}_p(L_p)}).\end{aligned}$$

A similar estimate for ψ in $H^{1/2}_p(L_p)$ has already been done in (6.4). For the term $\nabla\psi\lambda'(\psi)$ we will use (3.4). This leads to

$$|\nabla\psi\lambda'(\psi)|_{H^{1/2}_p(L_p)} \leq c(|\nabla\psi|_{H^{1/2}_p(L_p)} + |\nabla\psi|_{L_{\sigma'p}(L_{r'p})}|\psi|_{H^{1/2}_{\sigma p}(L_{rp})}),$$

since $\lambda' \in L_\infty(\mathbb{R})$. We will use the same strategy as in (6.4). First we observe that complex-interpolation and the Gagliardo-Nirenberg inequality lead to the desired estimate for $|\nabla\psi|_{H_p^{1/2}(L_p)}$. Secondly we make again use of the Gagliardo-Nirenberg inequality to obtain

$$|\nabla\psi|_{L_{\sigma' p}(L_{r'p})} \leq c|\psi|_{Z^1}^a |\psi|_{L_\infty(H_2^1)}^{1-a},$$

where $a = 1/\sigma' = (n/2 - n/r'p)/(3 + n/2 - n/p)$. Complex interpolation yields

$$|\psi|_{H_{\sigma p}^{1/2}(L_{rp})} \leq c|\psi|_{H_p^s(L_{rp})}^b |\psi|_{L_{\tau p}(L_{rp})}^{1-b}, \tag{6.7}$$

where $b = 1/2s$, $s = 1 - n/4r'p$ and

$$1/\sigma \geq b + (1-b)/\tau. \tag{6.8}$$

One more time the Gagliardo-Nirenberg inequality leads to the estimate

$$|\psi|_{L_{\tau p}(L_{rp})} \leq c|\psi|_{Z^1}^d |\psi|_{L_\infty(H_2^1)}^{1-d}, \tag{6.9}$$

with $d = 1/\tau = [n/2 - 1 - n/rp]_+/(3 - n/p + n/2)$. Finally we have to check if (6.8) is valid and if in this case the inequality is strict. We distinguish two cases. If $p \leq n$ we set $r' = 2n/p$ and if $p > n$ we set $r' = 2$ (then $r' \in [2,3]$). In the first case we have $s = 7/8$, $b = 4/7$ and $[n/2 - 1/2 - n/p]_+ = 0,\ n = 1,2,3$, since $p \geq 2$. Thus (6.8) is equivalent to the condition

$$p \geq \frac{6n}{25 - 4n}\ ,$$

which is always fulfilled and strict inequality holds if $n \leq 3$. In the second case we have $r' = 2$, thus $s = 1 - n/8p > 7/8$. We set $s = 7/8$ and therefore $b = 4/7$. Then (6.8) is equivalent to

$$18 - 4n + n/p \geq 6[n/2 - 1 - n/2p]_+.$$

This inequality is obviously fulfilled and additionally strict, if $n = 1, 2$. So let $n = 3$. Then $n/2 - 1 - n/2p > 0$ and we obtain $1 + 4/p > 0$, which is certainly true.

The next estimate will be done for the term $\partial_\nu(\lambda'(\psi)F(\psi))$ in Y_1. Again we use trace-theory to obtain

$$|\partial_\nu(\lambda'(\psi)F(\psi))|_{Y_1} \leq C(|\lambda'(\psi)F(\psi)|_{H_p^{1/2}(L_p)} + |\lambda'(\psi)F(\psi)|_{L_p(H_p^2)}).$$

The estimate of $\lambda'(\psi)F(\psi)$ in $L_p(H_p^2)$ has already been done. Making use of (3.4) and (3.6), with $s = 1/2$ and L_p instead of H_p^1, we obtain

$$|\lambda'(\psi)F(\psi)|_{H_p^{1/2}(L_p)} \leq C\left(1 + |\psi|_{H_p^{1/2}(L_p)} + (1 + |\psi|_{L_{\sigma' p}(L_{r'p})})(1 + |\psi|_{H_{\sigma p}^{1/2}(L_{rp})})\right),$$

since $\lambda', \lambda'' \in L_\infty(\mathbb{R})$. Therefore the estimate follows immediately from Step 1.

Step 3. Last but not least we have to consider $\Delta(\lambda'(\psi)\eta)$ in X and $\partial_\nu(\lambda'(\psi)\eta)$ in Y_1, where $\eta \in Z^2$ is a fixed function. We compute

$$\Delta(\lambda'(\psi)\eta) = \eta(\Delta\psi\lambda''(\psi) + |\nabla\psi|^2\lambda'''(\psi)) + 2\lambda''(\psi)\nabla\psi\nabla\eta + \lambda'(\psi)\Delta\eta.$$

Since $p \geq 2$ we have $Z^2 \hookrightarrow L_{3p}(J \times \Omega)$, hence the estimate for the first term follows from Step 2. Moreover by (6.6) we obtain

$$|\nabla\psi\nabla\eta|_{L_p(L_p)} \leq |\nabla\psi|_{L_{3p}(L_{3p})}|\nabla\eta|_{L_{3p/2}(L_{3p/2})} \leq c|\psi|^{\delta}_{Z^1(T)}|\psi|^{1-\delta}_{L_\infty(J;H^1_2(\Omega))}, \ \delta < 1,$$

since $\nabla\eta$ is a fixed function and

$$H^{1/2}_p(J; L_p(\Omega)) \cap L_p(J; H^1_p(\Omega)) \hookrightarrow L_{3p/2}(J \times \Omega).$$

Finally the last term $\lambda'(\psi)\Delta\eta$ is dominated by the fixed function $\Delta\eta \in L_p(J \times \Omega)$, since $\lambda' \in L_\infty(\mathbb{R})$. A last time we apply trace-theory to obtain

$$|\partial_\nu(\lambda'(\psi)\eta)|_{Y_1} \leq C(|\lambda'(\psi)\eta|_{H^{1/2}_p(L_p)} + |\lambda'(\psi)\eta|_{L_p(H^2_p)})$$

and then (3.4) leads to

$$|\lambda'(\psi)\eta|_{H^{1/2}_p(L_p)} \leq C(|\eta|_{H^{1/2}_p(L_p)} + |\eta|_{L_{\sigma' p}(L_{r'p})}|\psi|_{H^{1/2}_{\sigma p}(L_{rp})}).$$

Since $\eta \in L_\infty(J; H^1_p(\Omega))$ for all $p \geq 2$ we may choose $r' \in [2, 3]$, i.e., $r \in [3/2, 2]$ and σ' may be arbitrarily large. Then the claim follows from (6.7) and (6.9). The proof is complete.

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Numerical Approximation of PDEs and Clément's Interpolation

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In honor of Clément's retirement

Abstract. In this short paper, we present a formalism which specifies the notions of consistency and stability of finite element methods for the numerical approximation of nonlinear partial differential equations of elliptic and parabolic type. This formalism can be found in [4], [7], [10], and allows to establish a priori and a posteriori error estimates which can be used for the refinement of the mesh in adaptive finite element methods. In concrete cases, the Clément's interpolation technique [6] is very useful in order to establish local a posteriori error estimates. This paper uses some ideas of [10] and its main goal is to show in a very simple setting, the mathematical arguments which lead to the stability and convergence of Galerkin methods. The bibliography concerning this subject is very large and the references of this paper are no exhaustive character. In order to obtain a large bibliography on the a posteriori error estimates, we report the lecturer to Verfürth's book and its bibliography [12].

1. Introduction

Let X and Y be two reflexive Banach spaces (the norm is denoted by $\|.\|$ for both spaces when there is no ambiguity), and let

$$F : X \to Y'$$

be a C^1 mapping from X into the dual space $Y^{'}$ of Y. We are looking for a $u \in X$ satisfying

$$F(u) = 0. \tag{1.1}$$

Problem $F(u) = 0$ is equivalent to

$$\langle F(u); v\rangle = 0 \qquad \text{for all } v \in Y, \tag{1.2}$$

where $\langle \cdot;\cdot\rangle$ denotes the duality pairing $Y' - Y$.

Example 1.1. $X = Y = H_0^1(\Omega)$ where $\Omega \subset R^d$, $d = 1, 2$ or 3, Ω is open, bounded and *regular*, $Y' = H^{-1}(\Omega)$, and

$$F(u) = -\Delta u + u^3 - f,$$

where $f \in L^2(\Omega)$ is given.

In this example we have $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ and consequently we are looking for $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla u.\nabla v\, dx + \int_\Omega u^3 v\, dx = \int_\Omega f v\, dx \qquad \text{for all } v \in H_0^1(\Omega).$$

In order to give an approximation of problem (1.1) we choose two families of finite-dimensional subspaces $X_h \subset X$ and $Y_h \subset Y$, $0 < h \leq 1$, such that

for all $x \in X$ and for all $y \in Y$ we have

$$\lim_{h\to 0} \inf_{x_h \in X_h} \|x - x_h\| = 0 \qquad \text{and} \qquad \lim_{h\to 0} \inf_{y_h \in Y_h} \|y - y_h\| = 0. \tag{1.3}$$

In this case, the approximate problem consists in finding $u_h \in X_h$ satisfying

$$\langle F(u_h); v_h \rangle = 0 \qquad \text{for all } v_h \in Y_h. \tag{1.4}$$

If $\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_N$ is a basis of X_h and $\psi_1, \psi_2, \psi_3, \ldots, \psi_M$ is a basis of Y_h, we develop $u_h = \Sigma_{j=1}^N \alpha_j \varphi_j$ in order to obtain the finite-dimensional problem by looking for $\overrightarrow{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_N) \in \mathbb{R}^N$ satisfying the M following equations:

$$\left\langle F\left(\Sigma_{j=1}^N \alpha_j \varphi_j\right); \psi_k \right\rangle = 0, \qquad \text{for all } k = 1, 2, 3, \ldots, M. \tag{1.5}$$

Equations (1.4) or equivalently (1.5) are called *Petrov-Galerkin approximation of Equation* (1.1).

Example 1.2. We consider Example 1 in which we suppose that $\Omega \subset R^2$ is an open polygonal domain. If Υ_h is a triangulation of $\overline{\Omega}$ (Υ_h is a set of triangles K such that $\overline{\Omega} = \cup_{K\in\Upsilon_h} K$ and the intersection between two different triangles of Υ_h is either void or a vertex or a side shared by both), we can define the usual finite element subspaces

$$\begin{aligned} X_h = Y_h =_{def} \{v : \overline{\Omega} \to R,\ &\text{continuous with } v|_K \in \mathbb{P}_1(K), \\ &\forall K \in \Upsilon_h, v = 0 \text{ on } \partial\Omega\} \end{aligned} \tag{1.6}$$

where $\mathbb{P}_1(K)$ is the set of polynomial functions of degree ≤ 1 defined on the triangle K. The Petrov-Galerkin approximation of Equation (1.1) related to Example 1 consists in looking for $u_h \in X_h$ satisfying

$$\int_\Omega \left\{\nabla u_h.\nabla v_h + u_h^3 v_h - f v_h\right\} dx = 0 \qquad \forall v_h \in X_h.$$

From now, we will assume that Equation (1.1) possesses a solution $u \in X$ and we will answer to the main question: under which conditions we can find a solution u_h of Equation (1.4) close to u? Moreover, is it possible to estimate the error $\|u - u_h\|$?

In order to answer to these questions, we have to introduce two notions: the stability and the consistency of Petrov-Galerkin approximations.

2. Stability, consistency and convergence

In order to illustrate the questions of stability consistency and convergence, we look at Figure 1 on which we visualize the different mathematical symbols we are using in the following. In particular we denote by $DF(u) : X \to Y'$ the Fréchet derivative of F at point u, and $L(X, Y')$ the space of linear continuous mappings from X into Y'. In all the sequel, we will suppose that $DF : X \to L(X, Y')$ is a Lipschitzian mapping in u . The norm of $DF(u)$ in $L(X, Y')$ or the norm of the inverse $DF(u)^{-1}$ (if it exists) in $L(Y', X)$ is still denoted by $\|.\|$.

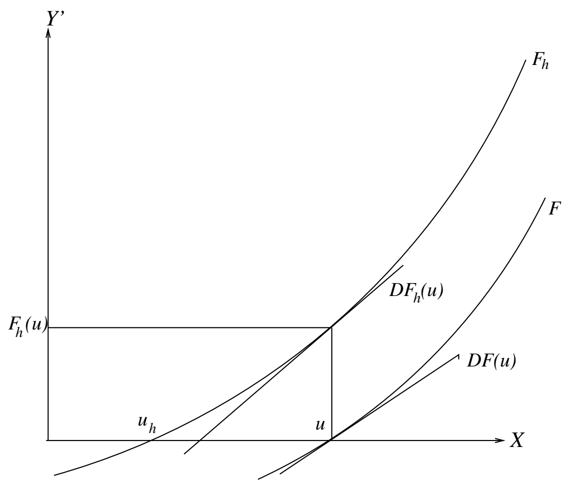

FIGURE 1. Setting of Theorem 1.

Let $F_h : X \to Y^{'}$, $0 \leq h \leq h_0$, be a family of C^1-mappings such that if an element $u_h \in X$ satisfies $F_h(u_h) = 0$, then necessarily $u_h \in X_h$ and u_h is solution of (1.4) or equivalently (1.5). We will see below how we can build such a mapping F_h.

Theorem 2.1. (see [10]) *Assume that u is a solution of Problem* (1) *and that*

(H1) $DF(u)$ *is an isomorphism from X onto Y',*

(H2) $\lim_{h\to 0} \|F_h(u)\| = 0$,

(H3) $\lim_{h\to 0} \|DF(u) - DF_h(u)\| = 0$, (H4) *there exist two positive constants L and ϵ such that*

$$\|DF_h(u) - DF_h(w)\| \leq L \|u - w\|$$

for all $w \in X$ with $\|u - w\| \leq \epsilon$.

Then:

(a) *There exist $h_0 > 0$ and a ball centered in u and with radius $\delta > 0$ denoted by $B(u, \delta) \subset X$ such that, for all $h \leq h_0$, there exists only one $u_h \in B(u, \delta)$ satisfying $F_h(u_h) = 0$.*
(b) *We have the error estimates:*

$$\|u - u_h\| \leq C \|F_h(u)\| \quad (\textit{a priori error, consistency error}); \tag{2.1}$$
$$\|u - u_h\| \leq C \|F(u_h)\| \quad (\textit{a posteriori error, residual error}); \tag{2.2}$$

where C is a constant independent of $h \leq h_0$.

Remark 2.2. Hypotheses (H1) and (H3) imply that for $h \leq h_0$ small enough, the operator $DF_h(u)$ is an isomorphism from X onto Y' and there exists a constant M such that $\left\|DF_h(u)^{-1}\right\| \leq M$ for all $h \leq h_0$. This property is called *stability*. Hypothesis (H2) is called *consistency* of the approximation.

Proof. We start by defining the operator $G : X \to X$ as

$$G(v) = v - DF_h(u)^{-1} F_h(v). \tag{2.3}$$

As we said in the above remark, there exist two constants M and h_0 such that $\left\|DF_h(u)^{-1}\right\| \leq M$, for all $h \leq h_0$ and consequently G is well defined. Moreover if u_h is a fixed point of G, i.e., $u_h = G(u_h)$, then u_h is a zero of F_h, i.e., $F_h(u_h) = 0$.

Let now $\delta \leq \epsilon$ (ϵ given in (H4)), and let v, w be two elements in the closed ball $B(u, \delta)$ centered in u with radius δ, i.e., $v, w \in X$ are such that $\|u - v\| \leq \delta$, and $\|u - w\| \leq \delta$. We verify easily the following relationship

$$\begin{aligned} G(v) &- G(w) \\ &= DF_h(u)^{-1} \left[\int_0^1 \left(DF_h(u) - DF_h(sv + (1-s)w)\right)(v - w)\, ds\right]. \end{aligned} \tag{2.4}$$

By using (H4), we obtain from (2.4) :

$$\|G(v) - G(w)\| \leq ML\delta \|v - w\|. \tag{2.5}$$

Now we choose $\delta \leq (2ML)^{-1}$ in order to obtain for all $v, w \in B(u, \delta)$ and $h \leq h_0$:

$$\|G(v) - G(w)\| \leq \frac{1}{2} \|v - w\|. \tag{2.6}$$

By using Hypothesis (H2), we can choose h_0 in order to have $\|F_h(u)\| \leq \delta/2M$ for $h \leq h_0$ and consequently we obtain for $v \in B(u,\delta)$:

$$\begin{aligned}\|u - G(v)\| &\leq \|u - G(u)\| + \|G(u) - G(v)\| \\ &\leq \left\|DF_h(u)^{-1}\right\| \|F_h(u)\| + \frac{1}{2}\|u - v\| \\ &\leq M\|F_h(u)\| + \frac{1}{2}\|u - v\| \leq \delta; \qquad (2.7)\end{aligned}$$

that proves that $G(v)$ belongs to $B(u,\delta)$ when v is in $B(u,\delta)$. The mapping G is a strict contraction of the ball $B(u,\delta)$ and so possesses a unique fixed point in $B(u,\delta)$ denoted by u_h. We can conclude that $F_h(u_h) = 0$ and so the first part (a) of the theorem is proven.

In order to prove the second part (b) we take $v = u_h$ in (2.7) by remarking that $u_h = G(u_h)$. We obtain the inequality (2.1) with $C = 2M$.

For obtaining the error estimate (2.2), we write

$$F(u_h) = F(u) + \int_0^1 DF(su_h + (1-s)u)(u_h - u)\,ds.$$

Since $F(u) = 0$, we have:

$$\begin{aligned}&\|u - u_h\| \\ &= \left\|DF(u)^{-1}\left[F(u_h) + \int_0^1 (DF(u) - DF(su_h + (1-s)u))(u_h - u)\,ds\right]\right\| \\ &\leq \left\|DF(u)^{-1}\right\|\left[\|F(u_h)\| + \max_{s\in[0,1]}\|DF(u) - DF(su_h + (1-s)u)\|\,\|u_h - u\|\right].\end{aligned}$$

By using Hypothesis (H2) together with (2.1), we have the convergence of u_h to u when h tends to zero and it suffices to take h_0 small enough in order to have for $h \leq h_0$:

$$\left\|DF(u)^{-1}\right\| \max_{s\in[0,1]}\|DF(u) - DF(su_h + (1-s)u)\| \leq 1/2.$$

In this case we conclude that $C = 2\left\|DF(u)^{-1}\right\|$ in (2.2). □

Remark 2.3. In (2.2), we can see that $C = \beta\left\|DF(u)^{-1}\right\|$ where $\beta > 1$ and β is close to 1 if h_0 is chosen small enough.

Moreover it is easy to show from the above relations that there exists a positive constant $\gamma \leq 1$ for which we have the a posteriori error estimate from below:

$$\|u_h - u\| \geq \gamma\left\|DF(u)^{-1}F(u_h)\right\| \geq \gamma\|DF(u)\|^{-1}\|F(u_h)\|. \qquad (2.8)$$

Inequalities (2.2) and (2.8) are called *a posteriori error estimate* because it exist two constants C_1 and C_2 (independent of h) such that the error $\|u - u_h\|$ satisfies, for $h \leq h_0$ small enough, $C_1\|F(u_h)\| \leq \|u - u_h\| \leq C_2\|F(u_h)\|$ and these upper and lower bound depend on u_h only. That means, in a lot of practical cases, that it is possible to give a bound for the error of the approximate solution u_h related to the exact solution u, when u_h is computed.

3. The Petrov-Galerkin situation

It is known that if X and Y are reflexive Banach spaces (see [1]), then Hypothesis (H1) of Theorem 1 is equivalent to the following inf-sup conditions:

$$\inf_{x\in X,\|x\|=1} \sup_{y\in Y,\|y\|=1} \langle DF(u)\,x;y\rangle > 0,$$

$$\sup_{x\in X,\|x\|=1} \langle DF(u)\,x;y\rangle > 0 \qquad \forall y \in Y, y \neq 0.$$

By assuming that these hypotheses are uniformly true in h on a discrete level, that is to say:

$$\inf_{x_h\in X_h,\|x_h\|=1} \sup_{y_h\in Y_h,\|y_h\|=1} \langle DF(u)\,x_h;y_h\rangle \geq \gamma > 0 \quad \text{for all } h, \tag{3.1}$$

where γ is a positive constant independent of h, and

$$\dim X_h = \dim Y_h, \tag{3.2}$$

we can define the projectors $Q_h : X \to X_h$ and $P_h : Y \to Y_h$ with the relations:

$$\langle DF(u)\,Q_h w;v_h\rangle = \langle DF(u)\,w;v_h\rangle, \qquad \forall v_h \in Y_h, \forall w \in X, \tag{3.3}$$

$$\langle DF(u)\,w_h;P_h v\rangle = \langle DF(u)\,w_h;v\rangle, \qquad \forall w_h \in X_h, \forall v \in Y. \tag{3.4}$$

In fact (3.1) and (3.2) imply that the matrix with coefficients $\langle DF(u)\,\varphi_j;\psi_k\rangle$, $1 \leq j,k \leq N = M$, is invertible.

Let now $F_h : X \to Y'$ be defined by

$$\langle F_h(w);v\rangle = \langle F(w);P_h v\rangle + \langle DF(u)\,w;(I-P_h)\,v\rangle \text{ for all } v \in Y, \forall w \in X. \tag{3.5}$$

Lemma 3.1. *The two following assertions are equivalent:*

(a) $u_h \in X_h$ *is solution of* (1.4)*;*
(b) $u_h \in X$ *is such that* $F_h(u_h) = 0$.

Proof. It is obvious that a) implies b). Let us show that b) implies a). To do this we suppose that $u_h \in X$ is such that $F_h(u_h) = 0$ and we take $v = y - P_h y$ in (3.5). We obtain

$$\begin{aligned}0 = \langle DF(u)\,u_h;y-P_h y\rangle &= \langle DF(u)\,u_h;y\rangle - \langle DF(u)\,Q_h u_h;P_h y\rangle \\ &= \langle DF(u)\,(I-Q_h)\,u_h;y\rangle.\end{aligned}$$

This relationship proves that $(I-Q_h)\,u_h = 0$ and consequently $u_h = Q_h u_h \in X_h$.

By taking $v = v_h \in Y_h$ in (3.5), we obtain $\langle F(u_h);v_h\rangle = 0$, that means that u_h is solution of (1.4). $\square$

With this setting, we obtain from Theorem 1:

Theorem 3.2. *Assume that u is a solution of Problem* (1.1) *and that $DF(u)$ is an isomorphism from X onto Y'. Moreover we assume Hypothesis* (1.3) *and Hypotheses* (3.1) *and* (3.2) *are satisfied. Then Problem* (1.4) *possesses a unique solution u_h in a neighborhood of u and we have the error estimates:*

$$\|u - u_h\| \leq C \inf_{v_h \in X_h} \|u - v_h\| \quad (\textit{a priori error, consistency error}); \tag{3.6}$$

$$\|u - u_h\| \leq C \|F(u_h)\| \quad (\textit{a posteriori error, residual error}). \tag{3.7}$$

Proof. It suffices to show that the mapping F_h build in (3.5) satisfies the hypotheses of Theorem 1. In order to verify (H2) we write:

$$\begin{aligned}\|F_h(u)\| &= \sup_{y \in Y, \|y\|=1} \langle F_h(u); y\rangle = \sup_{y \in Y, \|y\|=1} \langle DF(u)\, u; (I - P_h)\, y\rangle \\ &= \sup_{y \in Y, \|y\|=1} \langle DF(u)(u - Q_h u); y\rangle \leq \|DF(u)\| \, \|u - Q_h u\| .\end{aligned} \tag{3.8}$$

Now it is easy to verify by using (3.1) that, for all $x_h \in X_h$, we have:

$$\begin{aligned}\gamma \|x_h - Q_h u\| &\leq \sup_{y_h \in Y_h, \|y_h\|=1} \langle DF(u)(x_h - Q_h u); y_h\rangle \\ &= \sup_{y_h \in Y_h, \|y_h\|=1} \langle DF(u)(x_h - u); y_h\rangle \leq \|DF(u)\| \, \|x_h - u\| .\end{aligned}$$

It follows that

$$\|u - Q_h u\| \leq \|u - x_h\| + \|x_h - Q_h u\| \leq \left(1 + \frac{\|DF(u)\|}{\gamma}\right) \|u - x_h\| ,$$

and with (3.8), we finally obtain for all $x_h \in X_h$:

$$\|F_h(u)\| \leq \|DF(u)\| \left(1 + \frac{\|DF(u)\|}{\gamma}\right) \|u - x_h\| .$$

This last estimate proves that $\|F_h(u)\|$ tends to zero when h goes to zero (see (1.3)), and the consistency is proven.

Hypothesis (H3) of Theorem 1 is satisfied because it is easy to verify with the definition of F_h that $DF_h(u) = DF(u)$. Moreover, by using (3.1), we can verify that the operator P_h is uniformly bounded in h and taking into account the fact that DF is Lipschitzian in u, we immediately obtain Hypothesis (H4). □

4. Example and Clément interpolation

We consider the previous example where $\Omega \subset R^2$ is an open polygonal domain and $F(u) = -\Delta u + u^3 - f$ with given $f \in L^2(\Omega)$. In this case, it is known that there exists a solution $u \in H_0^1(\Omega) = X = Y$. Moreover if Ω is convex, then $u \in H^2(\Omega)$.

In this case we obtain:

$$DF(u)\,w = -\Delta w + 3u^2 w, \tag{4.1}$$

and consequently

$$b(w,v) \underset{\text{def}}{=} \langle DF(u)\,w; v\rangle = \int_\Omega \left(\nabla w.\nabla v + 3u^2 wv\right) dx \tag{4.2}$$

is a bilinear, continuous and coercive form on $X = H_0^1(\Omega)$. It follows that $DF(u): X \to Y' = H^{-1}(\Omega)$ is an isomorphism.

Let Υ_h be a regular triangulation of $\overline{\Omega}$ (in the sense of [5]) and remember that the usual finite element subspaces of $H_0^1(\Omega)$ with piecewise polynomial functions of degree 1 are given by

$$\begin{aligned} X_h \quad = \quad Y_h =_{def} \{ & v : \overline{\Omega} \to R,\ \text{continuous with}\ v|_K \in \mathbb{P}_1(K), \\ & \forall K \in \Upsilon_h, v = 0 \text{ on } \partial\Omega\} \end{aligned} \tag{4.3}$$

where $\mathbb{P}_1(K)$ is the set of polynomial functions of degree ≤ 1 defined on the triangle K. Denoting by $h = \max_{K\in\Upsilon_h} \operatorname{diam}(K)$, it is well known (see [5]) that Hypothesis (3) is satisfied when h tends to zero, and there exists a constant C (independent of h and v) such that

$$\inf_{v_h \in X_h} \|v - v_h\|_{1,\Omega} \leq Ch\,\|v\|_{2,\Omega}\,, \forall v \in H^2(\Omega) \cap H_0^1(\Omega)\,, \forall h \leq h_0.$$

Here the norm $\|.\|_{m,\Omega}$ denotes the $H^m(\Omega)-$norm, $m = 1, 2$.

Since the bilinear form $b(.,.)$ is coercive on $X = H_0^1(\Omega)$, Hypotheses (3.1) and (3.2) are automatically satisfied. It follows that we can apply Theorem 2 to obtain a unique $u_h \in X_h$ in a neighborhood of u (in fact in X_h) satisfying

$$\int_\Omega \left\{\nabla u_h.\nabla v_h + u_h^3 v_h - f v_h\right\} dx = 0 \qquad \forall v_h \in X_h. \tag{4.4}$$

The a priori error estimate (3.6) leads, when Ω is convex (i.e., when $u \in H^2(\Omega)$), to $\|u - u_h\|_{1,\Omega} \leq C\,\|u\|_{2,\Omega}\,h$, where C is a constant independent of the triangulation Υ_h and u. But in principle it is not obvious to give an estimation of $\|u\|_{2,\Omega}$ and, furthermore, this error estimate is non local. For these reasons we establish a posteriori error estimates depending on the approximate solution u_h which is locally computable.

To do this, we consider now the a posteriori error estimate (3.7) and we have to calculate:

$$\|F(u_h)\|_{-1,\Omega} = \sup_{v\in H_0^1(\Omega), \|v\|=1} \int_\Omega \left\{\nabla u_h.\nabla v + u_h^3 v - fv\right\} dx, \tag{4.5}$$

where $\|.\|_{-1,\Omega}$ denotes the dual $H^{-1}(\Omega)-$norm. In order to give a local error estimate, we successively use (4.4) and an integration by part for obtaining, for any $v \in H_0^1(\Omega), \|v\|_{1,\Omega} = 1$, and for any $v_h \in X_h$:

$$\begin{aligned}
&\int_\Omega \left\{ \nabla u_h . \nabla v + u_h^3 v - fv \right\} dx \\
&\quad = \sum_{K \in \Upsilon_h} \int_K \left\{ \nabla u_h . \nabla (v - v_h) + u_h^3 (v - v_h) - f(v - v_h) \right\} dx \\
&\quad = \sum_{K \in \Upsilon_h} \left[\int_K \left\{ -\Delta u_h (v - v_h) + u_h^3 (v - v_h) - f(v - v_h) \right\} dx \right. \\
&\qquad\qquad \left. + \int_{\partial K} \frac{\partial u_h}{\partial n} (v - v_h)\, ds \right] \\
&\quad \leq \sum_{K \in \Upsilon_h} \left[\left| \int_K \left\{ -\Delta u_h (v - v_h) + u_h^3 (v - v_h) - f(v - v_h) \right\} dx \right| \right. \\
&\qquad\qquad \left. + \frac{1}{2} \left| \int_{\partial K} \left[\frac{\partial u_h}{\partial n} \right] (v - v_h)\, ds \right| \right], \qquad (4.6)
\end{aligned}$$

where ∂K is the boundary of K, and $\left[\frac{\partial u_h}{\partial n}\right]$ denotes the jump of the normal derivative of u on ∂K when we have fixed a normal direction n on each internal side of the triangulation (with this notation we set $-\frac{1}{2}\left[\frac{\partial u_h}{\partial n}\right] = \frac{\partial u_h}{\partial n}$ on the sides of triangles which are on the boundary $\partial\Omega$ of Ω, where here n is the external normal vector). So we obtain, if $\|.\|_{0,T}$ denotes the L^2-norm on a set T:

$$\begin{aligned}
\|F(u_h)\|_{-1,\Omega} &= \sup_{v \in H_0^1(\Omega), \|v\| = 1} \int_\Omega \left\{ \nabla u_h . \nabla v + u_h^3 v - fv \right\} dx \\
&\leq \sup_{v \in H_0^1(\Omega), \|v\|_{1,\Omega} = 1} \inf_{v_h \in X_h} \sum_{K \in \Upsilon_h} \left(\left\| -\Delta u_h + u_h^3 - f \right\|_{0,K} \|v - v_h\|_{0,K} \right. \\
&\qquad\qquad \left. + \frac{1}{2} \left\| \left[\frac{\partial u_h}{\partial n} \right] \right\|_{0,\partial K} \|v - v_h\|_{0,\partial K} \right). \qquad (4.7)
\end{aligned}$$

The question now is how to choose v_h for obtaining a local estimation in (4.7)? It is not possible to choose the classical interpolation of v because $v \in H_0^1(\Omega)$ is not continuous when $d = 2$ or 3. This question can be solved by using the Clément's interpolation which is a local L_2-projection of v.

Let Ω_P be the set of all triangles having vertex $P \in \Omega$. The local L_2-projection of v, $R_p^0 : H^1(\Omega_p) \to \mathbb{P}_0(\Omega_p)$, is uniquely defined by $\int_{\Omega_p} \left(v - R_p^0 v\right) \varphi \, dx = 0$, $\forall \varphi \in \mathbb{P}_0(\Omega_p)$, where $\mathbb{P}_0(\Omega_p)$ denotes the constant functions on Ω_p. It is also possible to take other projections such that $R_p^1 : H^1(\Omega_p) \to \mathbb{P}_1(\Omega_p)$ defined by $\int_{\Omega_p} \left(v - R_p^1 v\right) \varphi \, dx = 0$, $\forall \varphi \in \mathbb{P}_1(\Omega_p)$, or $R_p : H^1(\Omega_p) \to X_p$ defined by $\int_{\Omega_p} (v - R_p v) \varphi \, dx = 0$, $\forall \varphi \in X_p$ where $X_p = \big\{ w \in H^1(\Omega_p) : w_{/K} \in \mathbb{P}_1(K)$, $\forall K \subset \bar{\Omega}_p \big\}$. With these notations we can define $v_h \in X_h$ by

$$\begin{array}{ll}
v_h(P) = \left(R_p^0 v\right)(P), & \forall P = \text{ vertex of } \gamma_h \text{ in } \Omega, \\
v_h(P) = 0, & \forall P = \text{ vertex of } \gamma_h \text{ on } \partial\Omega.
\end{array}$$

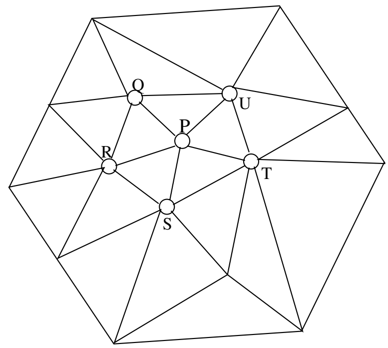

FIGURE 2. The boundary $QRSTU$ of Ω_P.

In [6], Clément proves that if the triangulation is regular, then there exists a constant C such that:

$$\|v - v_h\|_{0,K} \leq Ch_K \|v\|_{1,\Lambda_K} \qquad \text{for all } v \in H_0^1(\Omega), \tag{4.8}$$

where $\Lambda_K = \{T \in \Upsilon_h : T \cap K \neq \phi\}$, and h_K is the diameter of triangle K. With this estimate we can prove (see [12] for instance) that:

$$\|v - v_h\|_{0,l} \leq C\sqrt{h_\ell} \|v\|_{1,\Lambda_K} \qquad \text{for all side } \ell \text{ of } K, \tag{4.9}$$

where h_l is the length of ℓ.

Remark 4.1. Interpolation of nonsmooth functions is fundamental in finite element a posteriori error analysis. At the origin, Clément in [6] defined an optimal order interpolation operator which does not a priori preserve the boundary conditions. In [11], Scott and Zhang propose another type of interpolation of nonsmooth functions which naturally satisfy the boundary conditions. Several authors have proposed variant versions of Clément's operator and we don't want to be exhaustive character. However we just mention the generalization of Clément's interpolation to isoparametric finite elements (see [2], [3]) or to anisotropic finite elements (see [8]).

Estimates (4.7), (4.8), (4.9) and Theorem 2 lead to the following a posteriori error estimate:

Theorem 4.2. *If the triangulation Υ_h is regular, then there exists a constant C independent of the mesh such that:*

$$\|u - u_h\|_{1,\Omega} \leq C(\sum_{K\in\Upsilon_h} \eta(K)^2)^{\frac{1}{2}}, \tag{4.10}$$

where $\eta(K)$ is the local estimator defined by

$$\eta(K) = h_K \left\|-\Delta u_h + u_h^3 - f\right\|_{0,K} + \sum_{l=1}^{3} \sqrt{h_{l,K}} \left\| \left[\frac{\partial u_h}{\partial n} \right] \right\|_{0,\partial K_l}. \tag{4.11}$$

Here $h_{l,K}$ denotes the length of the side ∂K_l of $K \in \Upsilon_h$, $l = 1, 2, 3$.

Remark 4.3. It is possible (but more complicated) to establish a lower estimation of type $(\sum_{K\in\Upsilon_h}\eta(K)^2)^{\frac{1}{2}} \leq \widetilde{C}\,\|u-u_h\|_{1,\Omega}$ by using (2.8) (see [12] for instance). In this case we say that the error $\|u-u_h\|_{1,\Omega}$ is equivalent to the estimator $(\sum_{K\in\Upsilon_h}\eta(K)^2)^{\frac{1}{2}}$.

The estimator defined by (4.11) is only depending on the residual of the discrete solution u_h and on the triangle K. Clearly, since we have chosen to take piecewise polynomial functions of degree 1 for defining X_h, we will have $\Delta u_h = 0$ on K. In practice, if we want to use the a posteriori error estimate (4.10) in order to build a good mesh Υ_h (adaptive finite element method), we start by computing a solution u_h on some triangulation and we split all the triangles K for which $\eta(K)$ is too large or, at the opposite, we suppress a triangle when $\eta(K)$ is small enough by following some procedure (see [9] for instance).

5. A particular case

In this section we want to discuss the frequent case where $Y = X$ as in Example 1 and $F : X \to X'$ has the form

$$F(u) = Lu + N(u) \tag{5.1}$$

where $L \in L(X, X')$ and $N : X \to Z$ is a C^1 mapping from X onto a Banach space $Z \subset X'$. We assume that

$$\exists\alpha > 0 \text{ such that } \langle Lx; x\rangle \geq \alpha\,\|x\|^2 \qquad \forall x \in X, \tag{5.2}$$

$$\text{and the embedding } Z \subset X' \text{ is compact.} \tag{5.3}$$

Theorem 5.1. *Assume Hypotheses* (1.3), (5.2) *and* (5.3) *and let* $u \in X$ *be a solution of* $F(u) = 0$. *Then if* $DF(u)$ *is an isomorphism from* X *onto* X', *the approximate problem* (1.4) (*with* $Y_h = X_h$) *possesses a unique solution* u_h *in a neighborhood of* u *and we have the error estimates:*

$$\|u-u_h\| \leq C \inf_{v_h\in X_h}\|u-v_h\| \quad (\textit{a priori error, consistency error}); \tag{5.4}$$

$$\|u-u_h\| \leq C\,\|F(u_h)\| \quad (\textit{a posteriori error, residual error}). \tag{5.5}$$

Proof. By defining the continuous bilinear form $a(x;y) = \langle Lx; y\rangle$, $x, y \in X$, we remark that Hypothesis (5.2) implies that a is coercive and by using Lax-Milgram Lemma, it is easy to show that:

a) L is an isomorphism from X onto X',
b) There exist two projectors R_h and $R_h^* : X \to X_h$ satisfying $a(R_h x; y_h) = a(x; y_h)$ for all $y_h \in X_h$, $x \in X$, and $a(x_h; R_h^* y) = a(x_h; y)$ for all $x_h \in X_h$, $y \in X$. Moreover these two projectors tend strongly to the identity in X (i.e., $\lim_{h\to 0}\|x - R_h x\| = \lim_{h\to 0}\|x - R_h^* x\| = 0\ \forall x \in X$) and are uniformly bounded for $h \leq h_0$.

In order to prove Theorem 4, it suffices (see Theorem 2) to show that there exists a constant γ such that

$$\inf_{x_h\in X_h,\|x_h\|=1}\ \sup_{y_h\in Y_h,\|y_h\|=1}\langle DF(u)\,x_h;y_h\rangle\geq\gamma>0\quad\text{for all } h. \tag{5.6}$$

To do this, we take $x_h\in X_h$ with $\|x_h\|=1$ and we define

$$y_h=R_h^*\left(I+L^{-1}DN(u)\right)x_h$$

where I is the identity in X and $DN(u)$ is the derivative of N at point u. We have:

$$\begin{aligned}\langle DF(u)\,x_h;y_h\rangle&=a\left(\left(I+L^{-1}DN(u)\right)x_h;y_h\right)\\&=a\left(\left(I+L^{-1}DN(u)\right)x_h;R_h^*\left(I+L^{-1}DN(u)\right)x_h\right)\\&=a\left(R_h\left(I+L^{-1}DN(u)\right)x_h;\left(I+L^{-1}DN(u)\right)x_h\right)\\&=a\left(\left(I+L^{-1}DN(u)\right)x_h;\left(I+L^{-1}DN(u)\right)x_h\right)\\&\qquad+\ a\left((R_h-I)\,L^{-1}DN(u)\,x_h;\left(I+L^{-1}DN(u)\right)x_h\right)\\&\geq\alpha\left\|\left(I+L^{-1}DN(u)\right)x_h\right\|^2\\&\qquad-\|L\|\left\|(R_h-I)\,L^{-1}DN(u)\right\|\left\|\left(I+L^{-1}DN(u)\right)\right\|.\end{aligned} \tag{5.7}$$

Since the operator $\left(I+L^{-1}DN(u)\right)$ is an isomorphism of X and $\|x_h\|=1$, there exists a constant $\eta>0$ such that $\left\|\left(I+L^{-1}DN(u)\right)x_h\right\|\geq\eta$. The operator $L^{-1}DN(u):X\to X$ is compact (see (5.3)) and R_h converges strongly to I when h tends to zero when h tends to zero. It is well known that these two properties imply the uniform convergence of $(R_h-I)\,L^{-1}DN(u)$ to zero. So we can choose h_0 in such a way $\|L\|\left\|(R_h-I)\,L^{-1}DN(u)\right\|\left\|\left(I+L^{-1}DN(u)\right)\right\|\leq\frac{1}{2}\alpha\eta^2$ for all $h\leq h_0$. Using (5.7), we obtain for all $x_h\in X_h$, $\|x_h\|=1$ and for all $h\leq h_0$:

$$\langle DF(u)\,x_h;y_h\rangle\geq\frac{1}{2}\alpha\eta^2,\qquad\text{where } y_h=R_h^*\left(I+L^{-1}DN(u)\right)x_h. \tag{5.8}$$

Since $\|y_h\|\leq\|R_h^*\|\left\|I+L^{-1}DN(u)\right\|$ and R_h^* is uniformly bounded when $h\leq h_0$, we finally obtain the existence of a constant $\gamma>0$ satisfying:

$$\sup_{y_h\in Y_h,\|y_h\|=1}\langle DF(u)\,x_h;y_h\rangle\geq\gamma, \tag{5.9}$$

that proves our theorem. □

Remark 5.2. We can replace (5.2) by

$$\inf_{\|x\|=1}\ \sup_{\|y\|=1}\langle Lx;y\rangle\geq\alpha\quad\forall x,y\in X, \tag{5.10}$$

$$\inf_{\|y\|=1}\ \sup_{\|x\|=1}\langle Lx;y\rangle\geq\alpha\quad\forall x,y\in X,$$

$$\inf_{\|x_h\|=1}\ \sup_{\|y_h\|=1}\langle Lx_h;y_h\rangle\geq\alpha\quad\forall x_h,y_h\in X_h, \tag{5.11}$$

for obtaining the same conclusion as in the Theorem 4.

It is clear that it is not necessary to take x and y in the same space in (5.10) and (5.11). We can chose another space Y for the variable y.

As an example we can consider the stationary Navier-Stokes problem:

$$\begin{array}{rcllr} -\triangle u + (u.\nabla)\, u + grad\ p & = & f & \text{in} & \Omega \subset R^d,\ d = 2, 3, \\ div\ u & = & 0 & \text{in} & \Omega, \end{array}$$

where the velocity u and the pressure p are the unknown with $u = 0$ on the boundary $\partial\Omega$ of Ω, and f is a given force field. In this example we set

- $X = H_0^1(\Omega)^d \times L_0^2(\Omega)$ where $L_0^2(\Omega) = \left\{ v \in L^2(\Omega) : \int_\Omega v dx = 0 \right\}$,
- $\langle L\mathbf{u}; \mathbf{v} \rangle = \int_\Omega (\nabla u : \nabla v - p\ div\ v - q\ div\ u)\, dx$, with $\mathbf{u} = (u, p)$ and $\mathbf{v} = (v, q)$,
- $\langle N(\mathbf{u}); \mathbf{v} \rangle = \int_\Omega ((u.\nabla)\, u.v - fv)\, dx$.

In this case the operator L corresponds to the Stokes problem and it is well known that (5.10) is true. If we set $Z = L^{\frac{4}{3}}(\Omega)^d$, then property (5.3) is true with $d = 2$ or 3. In order to apply Theorem 4, it suffices to build finite element spaces satisfying (5.11). An example when $d = 2$ is given by the usually called *P1isoP2* elements for u and P1 element for p. This example is treated in [4].

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On Prohorov's Criterion for Projective Limits

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A Philippe Clément, avec salutations amicales.

Abstract. We propose a modification of Prohorov's theorem on projective limits of Radon measures which can be directly applied to the construction of Wiener measure on the space of continuous functions, and we give such a construction.[1]

1. Introduction

When constructing stochastic processes, or rather the measures on spaces of paths corresponding to such processes, the theorem of Prohorov [5], or the generalized version of Prohorov's theorem due to Bourbaki [1, §4, Theorem 1], is extremely useful. It states exactly under which condition a consistent family of Radon probability measures has a projective limit. Therefore it is somewhat of a disappointment to see that in Bourbaki's work [1, §6.7], when it comes to constructing Wiener measure, an indirect route is chosen, via a construction of an auxiliary space involving a countable subset of the time interval, allowing a reduction to the case of countable projective limits. In Prohorov's elegant theorem it is merely required to choose a compact set K in the path space, and then to verify that the finite-dimensional marginals are carried, up to $\varepsilon > 0$, by the projections of this set. But in the case of Wiener measure, if one takes a simple compact set suggested by Ascoli's theorem, it is not clear what precisely are the images under finite-dimensional evaluations, and the verification of Prohorov's condition is not obvious. The main purpose of this paper is to show that a modification of Prohorov's theorem is possible, in which one starts with given finite-dimensional compact sets, and one then builds, as projective limit, an appropriate compact set K in path space, which implies the existence of the projective limit as in Prohorov's theorem. The finite-dimensional images of K may be smaller than the given finite-dimensional sets however, and there is no guarantee a priori that Prohorov's condition is satisfied

[1]First presented in Séminaire Clément, Delft, 1997/98.

by K. Since the compact set in Prohorov's theorem is in fact the projective limit of its finite-dimensional projections, Prohorov's theorem is a corollary of the theorem we suggest.

One consequence of the Prohorov theorem as formulated by Bourbaki, is the existence without further conditions, of projective limits of Radon measures when the index set is the set $\mathbb{N}$ of natural numbers [1, §4, Theorem 2]. If we apply the modified version of Prohorov's theorem to this situation, the proof of the existence of projective limits indexed by $\mathbb{N}$ becomes very simple.

The theorem on countable projective limits immediately gives the celebrated Kolmogorov extension theorem, for projective limits on arbitrary products, the sets in the Kolmogorov σ-algebra depending on a countable number of coordinates only (for historical details see [1, Note Historique]).

The main example we consider, which motivated the modified Prohorov Theorem, is the construction of Wiener measure. To see how the modified Prohorov Theorem works we give the complete construction of Wiener measure, which beside our theorem on projective limits only requires some elementary lemmas on finite-dimensional random vectors. If everything is put together we get a construction of Wiener measure close to that suggested by Nelson [3].

There is a technical point we have not been able to resolve. In the case of Prohorov's theorem there are various proofs. For instance L. Schwartz [6, p. 74–82] gives three proofs. One of these makes the assumption that the spaces are completely regular, and then, with the help of the Stone-Čech compactifications, reduces the matter to the case of compact spaces. In the present paper we make use of this method and in Theorem 4.1 we assume all spaces to be completely regular. All subsets of topological vector spaces and all metrizable spaces being completely regular, this is no great inconvenience, but it would be more satisfactory to have a proof which does not appeal the the Stone-Čech compactifications.

2. Bounded Radon measures

Let X be a topological Hausdorff space. Let $\mathcal{K} = \mathcal{K}(X)$ be the set of compact subsets of X. Let $\mathcal{B}(X)$ be the Borel σ-algebra, i.e., the smallest σ-algebra containing the set $\mathcal{O} = \mathcal{O}(X)$ of open subsets of X.

Definition 2.1. *A bounded positive Radon measure is a measure* $\mu : \mathcal{B}(X) \longrightarrow [0, +\infty)$ *which is inner regular with respect to compact sets:*

$$\mu(A) = \sup_{K \subset A} \mu(K), \qquad (K \text{ compact}). \tag{2.1}$$

By passing to complements (2.1) implies outer regularity:

$$\mu(A) = \inf_{A \subset O} \mu(O) \qquad (O \text{ open}). \tag{2.2}$$

Proposition 2.2.

1. *Let $\{O_i\}_{i\in I}$ be a family of open sets directed upwards (i.e., for $i,j \in I$ there is $k \in I$ such that $O_i \subset O_k$ and $O_j \subset O_k$.) Then*

$$\mu(\underset{i\in I}{\cup} O_i) = \sup_{i\in I} \mu(O_i). \tag{2.3}$$

2. *Let $\{F_i\}_{i\in I}$ be a family of closed sets directed downwards. Then*

$$\mu(\underset{i\in I}{\cap} F_i) = \inf_{i\in I} \mu(F_i). \tag{2.4}$$

Proof. Let $O = \cup_{i\in I} O_i$ and let $\lambda < \mu(O)$. There exists a compact set $K \subset O$ such that $\lambda \leq \mu(K)$. By the compactness of K there exists $i \in I$ such that $K \subset O_i$ and consequently $\lambda \leq \mu(K) \leq \mu(O_i)$. This implies $\mu(O) \leq \sup_{i\in I} \mu(O_i)$. The reverse inequality is obvious. Part 2. is obtained by taking complements. □

This part 2. is particularly useful in probability theory. If P is a Radon probability on $\mathbb{R}^{[0,T]}$ we have , if $a, b \in \mathbb{R}$

$$P\{a \leq x(t) \leq b \quad \forall\, t \in [0,T]\} = \inf_{\sigma} P\{a \leq x(t_i) \leq b, i = 1, \ldots, n\}$$

where $\sigma = \{t_1, t_2, \ldots, t_n\}$ is a finite subset of $[0,T]$.

Given a bounded Radon measure μ on X, a subset $A \subset X$ is μ-measurable if it belongs to the Lebesgue completion of $\mathcal{B}(X)$ with respect to μ, i.e., there exist Borel sets A' and A'' such that $A' \subset A \subset A''$ and $\mu(A'' \setminus A') = 0$. One then defines $\mu(A) = \mu(A') = \mu(A'')$.

Strong concentration

Let X and Y be Hausdorff spaces, and let $\pi : X \longrightarrow Y$ be a continuous map. Then, if μ is a bounded Radon measure on X, the image measure $\nu = \pi(\mu)$, defined by

$$\nu(A) = \mu(\pi^{-1}(A)), \quad A \in \mathcal{B}(Y) \tag{2.5}$$

is a bounded Radon measure on Y.

Now we consider the particular case in which X is contained, with continuous inclusion i, in a second Hausdorff space Y

$$X \underset{i}{\hookrightarrow} Y.$$

Let μ and ν be bounded Radon measures on X respectively Y. Then if $\nu = i(\mu)$ we have

$$\nu(A) = \mu(i^{-1}(A)) = \mu(A \cap X) \tag{2.6}$$

for all sets $A \in \mathcal{B}(Y)$. In particular

$$\nu(Y) = \mu(X). \tag{2.7}$$

Definition 2.3. *A bounded Radon measure ν on Y will be said to be* strongly concentrated *on X if there exists a bounded Radon measure μ on X such that $\nu = i(\mu)$.*

Proposition 2.4. *A Radon probability measure ν on Y is strongly concentrated on X if and only if*

$$\sup_{K \in \mathcal{K}(X)} \nu(K) = 1. \tag{2.8}$$

In that case there exists a unique Radon measure μ on X such that $\nu = i(\mu)$.

The nontrivial statement is a particular case of a result of L. Schwartz [6, Ch. I, Theorem 12, p. 39].

Corollary 2.5. *If ν is strongly concentrated on X, then ν is concentrated on X, i.e., X is ν-measurable and $\nu(Y \setminus X) = 0$.*

3. Prohorov's Theorem

Let $\mathfrak{S} = \{\sigma\}$ be an index set, ordered and increasingly directed, and let $\{X_\sigma, \pi_{\sigma\sigma'}\}$ be a projective system of topological Hausdorff spaces with continuous maps

$$\pi_{\sigma\sigma'} : X_{\sigma'} \longrightarrow X_\sigma \quad \sigma \leq \sigma' \tag{3.1}$$

such that for $\sigma \leq \sigma' \leq \sigma''$ we have

$$\pi_{\sigma\sigma'} \circ \pi_{\sigma'\sigma''} = \pi_{\sigma\sigma''}.$$

We recall that the projective limit, $\varprojlim X_\sigma$, is defined as the subset of the cartesian product $\prod X_\sigma$, composed of the coherent families $(x_\sigma)_{\sigma \in \mathfrak{S}}$, i.e., families such that

$$\pi_{\sigma\sigma'} x_{\sigma'} = x_\sigma \qquad \sigma \leq \sigma'.$$

Let X be a Hausdorff space, and assume continuous maps $\pi_\sigma : X \longrightarrow X_\sigma$ are given, satisfying the conditions

$$\pi_{\sigma\sigma'} \pi_{\sigma'} = \pi_\sigma \qquad \sigma \leq \sigma'$$

and separating the points of X. Then the map $x \mapsto (\pi_\sigma x)_\sigma$ is a continuous embedding

$$X \hookrightarrow \varprojlim_\sigma X_\sigma$$

by means of which we identify X as a subset of the projective limit. The topology of X is in general stronger than the topology induced by the projective limit.

Now assume given a system of Radon probabilities μ_σ, defined respectively on X_σ, which is compatible (or consistent), i.e., such that the image measures satisfy the conditions

$$\mu_\sigma = \pi_{\sigma\sigma'}(\mu_{\sigma'}) \qquad \sigma \leq \sigma'.$$

Theorem 3.1. (Prohorov, Bourbaki). *Let $(X_\sigma, \pi_{\sigma\sigma'}, \mu_\sigma)$ be a compatible system of Radon probability measures. Then there exists a Radon probability μ on X such that*

$$\pi_\sigma(\mu) = \mu_\sigma \qquad \sigma \in \mathfrak{S}$$

if and only if for every $\varepsilon > 0$ there exists a set $K \in \mathcal{K}(X)$ such that, we have

$$\mu_\sigma(\pi_\sigma(K)) \geq 1 - \varepsilon \qquad \sigma \in \mathfrak{S} \tag{$*$}$$

In that case μ is uniquely determined by the measures μ_σ, namely

$$\mu(K) = \inf_\sigma \mu_\sigma(\pi_\sigma(K)), \quad (K \text{ compact})$$

(cf. Prohorov [5], Bourbaki [1, §4.2 Theorem 1]).

If $X = \varprojlim X_\sigma$ the measure μ is called the *projective limit* of the measures μ_σ, and denoted $\varprojlim_\sigma \mu_\sigma$. If X is continuously embedded in $\varprojlim_\sigma X_\sigma$ and the conditions of the above theorem are satisfied, we say that the projective limit exists on X.

4. A modification of Prohorov's Theorem

The Prohorov Theorem can be improved on as follows:

Theorem 4.1. *Let $(X_\sigma, \pi_{\sigma\sigma'}, \mu_\sigma)$ be a compatible system of Radon probability measures. Assume that for every $\varepsilon > 0$ there exists a family of compact sets $K_\sigma \subset X_\sigma$ satisfying the following three conditions:*

a) $\pi_{\sigma\sigma'}(K_{\sigma'}) \subset K_\sigma \qquad \sigma \le \sigma'$,
b) *The projective limit, $K = \varprojlim K_\sigma$, is a compact subset of X,*
c) $\mu_\sigma(K_\sigma) \ge 1 - \varepsilon \qquad \sigma \in \mathfrak{S}$.

Then there exists a unique Radon probability μ on X such that $\pi_\sigma(\mu) = \mu_\sigma$ for all σ. Briefly, the projective limit $\mu = \varprojlim_\sigma \mu_\sigma$ exists on X.

Note that by condition a) we have maps $\pi_{\sigma\sigma'} : K_{\sigma'} \longrightarrow K_\sigma$, restrictions of the maps (3.1), so that the projective limit makes sense, and is a priori a compact subset of $\varprojlim X_\sigma$.

It is enough to assume that K_σ is compact from a certain point on, i.e., for $\sigma \ge \sigma_0$.

Theorem 3.1 is a corollary of Theorem 4.1. If $K \subset X$is a compact subset satisfying condition $(*)$ of Theorem 3.1 the sets $K_\sigma = \pi_\sigma(K)$ satisfy the assumptions a, b, c of Theorem 4.1: $K_\sigma = \pi_{\sigma\sigma'}(K_{\sigma'})$ if $\sigma \le \sigma'$, $K = \varprojlim K_\sigma$ is compact in X [2, §4], and $\mu_\sigma(K_\sigma) \ge 1 - \varepsilon$. Consequently, if for every $\varepsilon > 0$ a compact $K \subset X$ can be found satisfying the condition $(*)$, the projective limit $\mu = \varprojlim_\sigma \mu_\sigma$ exists on X.

Proof of Theorem 4.1. (Under the assumption that the spaces X_σ are completely regular.) They then have compactifications $\widehat{X}_\sigma$, e.g., the Stone-Čech compactifications, such that the maps $\pi_{\sigma\sigma'}$ have continuous extensions $\widehat{\pi}_{\sigma\sigma'} : \widehat{X}_{\sigma'} \longrightarrow \widehat{X}_\sigma$, which necessarily still form a compatible system. Let $\widehat{X} = \varprojlim \widehat{X}_\sigma$. Then this is a compact space. Let $\widehat{\mu}_\sigma$ equal the image of μ_σ under the injection, $X_\sigma \subset \widehat{X}_\sigma$. Then we have

$$\widehat{\pi}_{\sigma\sigma'}(\widehat{\mu}_{\sigma'}) = \widehat{\mu}_\sigma, \qquad \sigma \le \sigma'$$

and the projective limit $\widehat{\mu} = \varprojlim \widehat{\mu}_\sigma$ exists by an application of the Riesz-Markov theorem (Nelson [4]). Now, by definition of K, an element $x \in \widehat{X}$ belongs to K if and only if $\widehat{\pi}_\sigma(x)$ belongs to K_σ for all σ. Thus

$$K = \bigcap_\sigma \widehat{\pi}_\sigma^{-1}(K_\sigma).$$

Moreover the sets $F_\sigma = \widehat{\pi}_\sigma^{-1}(K_\sigma)$, closed in $\widehat{X}$, are directed downwards. In fact, by a) we have $K_{\sigma'} \subset \widehat{\pi}_{\sigma\sigma'}^{-1}(K_\sigma)$ so, if $\sigma \leq \sigma'$, $F_{\sigma'} = \widehat{\pi}_{\sigma'}^{-1}(K_{\sigma'}) \subset \widehat{\pi}_{\sigma'}^{-1}\widehat{\pi}_{\sigma\sigma'}^{-1}(K_\sigma) = \widehat{\pi}_\sigma^{-1}(K_\sigma) = F_\sigma$. It follows that $\widehat{\mu}(K) = \inf_\sigma \widehat{\mu}(\widehat{\pi}_\sigma^{-1}(K_\sigma)) = \inf_\sigma \widehat{\mu}_\sigma(K_\sigma) = \inf_\sigma \mu_\sigma(K_\sigma) \geq 1 - \varepsilon$, by c). By b) this implies that $\widehat{\mu}$ is strongly concentrated on X. If μ is the Radon measure on X who's image is $\widehat{\mu}$, then μ is the projective limit on X of the measures μ_σ . In fact, if H is a compact subset of X_σ, we have $\pi_\sigma(\mu)(H) = \mu(\pi_\sigma^{-1}(H)) = \mu(X \cap \widehat{\pi}_\sigma^{-1}(H)) = \widehat{\mu}(\widehat{\pi}^{-1}(H)) = \widehat{\mu}_\sigma(H) = \mu_\sigma(H)$. Therefore $\pi_\sigma(\mu) = \mu_\sigma$. □

4.1. Countable projective limits

Let $(X_n, n \in \mathbb{N}, \pi_{n,n+1})$ be a projective sequence of Hausdorff spaces, with continuous maps $\pi_{n,n+1} : X_{n+1} \longrightarrow X_n$. Let $\{\mu_n\}_{n\in\mathbb{N}}$ a projective sequence of Radon probabilities on these spaces, i.e., μ_n on X_n such that

$$\mu_n = \pi_{n,n+1}(\mu_{n+1}) \qquad n \in \mathbb{N}. \tag{4.1}$$

Let $X = \varprojlim X_n$ be the projective limit. Then it is known that the projective limit $\mu = \varprojlim \mu_n$ always exists. The proof of this fact, using Theorem 4.1, is as follows. Let $\varepsilon > 0$ be given. Let $K_1 \subset X_1$ be a compact subset such that $\mu_1(K_1) > 1 - \varepsilon$. We inductively construct a sequence of compact sets $K_n \subset X_n$ such that

$$\pi_{n,n+1}(K_{n+1}) \subset K_n, \qquad \mu_{n+1}(K_{n+1}) > 1 - \varepsilon \qquad n \in \mathbb{N}. \tag{4.2}$$

Assume the construction of $K_1, \ldots, K_n$ done. Since $\mu_n = \pi_{n,n+1}(\mu_{n+1})$ we have $\mu_n(K_n) = \mu_{n+1}(\pi_{n,n+1}^{-1}(K_n)) > 1-\varepsilon$. By the inner regularity of μ_{n+1} there exists a compact subset $K_{n+1} \subset \pi_{n,n+1}^{-1}(K_n)$ such that $\mu_{n+1}(K_{n+1}) > 1-\varepsilon$. The induction is complete. Since $K = \varprojlim K_n$ is compact, Theorem 4.1 implies that $\mu = \varprojlim \mu_n$ exists. Moreover $\mu(K) \geq 1 - \varepsilon$. (cf. [1, §4.3 Theorem 2]).

Corollary 4.2. (Kolmogorov Extension Theorem.) *Let $X = \prod_{t\in I} X_t$ be an arbitrary product of completely regular spaces. For $J \subset I$, finite, let μ_J be a Radon Probability on the product $X_J = \prod_{t\in J} X_t$ such that for $J_1 \subset J_2$, μ_{J_1} is the projection of μ_{J_2}. Then the projective limit m exists on the Kolomogorov σ-algebra κ.*

Proof. The projective limits μ_J exist for $J \subset I$ countable. The sets in κ being cylinder sets of the form $A \times \prod_{t\notin J} X_t$, with A a Borel set in $\prod_{t\in J} X_t$, it is sufficient to put $m(A) = \mu_J(A)$. Since countably many sets in κ can be written in this way with a fixed set J, the countable additivity of m follows. □

5. Application to Wiener measure

We now consider the case of Wiener measure on the space $C[0,T]$, of real-valued continuous functions on the interval $[0,T]$, equipped with the topology of uniform convergence. Let

$$X = C_0[0,T] = \{x \in C[0,T] : x(0) = 0\}.$$

We define $\mathfrak{S}$ to be the set of finite subdivisions

$$\sigma = \{t_0, t_1, \ldots, t_n\} \qquad 0 \le t_0 < t_1 < \cdots < t_n \le T \tag{5.1}$$

ordered by inclusion. We take $X_\sigma = \mathbb{R}^\sigma$ and let

$$\pi_\sigma : X \longrightarrow X_\sigma, \quad x \mapsto (x(t_0), \ldots, x(t_n))$$

denote the restriction or evaluation map. If $\sigma \le \sigma'$ similarly

$$\pi_{\sigma\sigma'} : X_{\sigma'} \longrightarrow X_\sigma$$

denotes the restriction map or projection.

For $t > 0$ let

$$K_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}. \tag{5.2}$$

Then the marginal measure corresponding to $\sigma = \{t_1, t_2, \ldots, t_n\}$, with $t_1 > 0$, is the Gaussian probability measure on $\mathbb{R}^{\{t_1,\ldots,t_n\}}$

$$P_\sigma(dx) = K_{t_n - t_{n-t}}(x_n - x_{n-1}) \ldots K_{t_2 - t_1}(x_2 - x_1) K_{t_1}(x_1) dx_n \ldots dx_1. \tag{5.3}$$

If $t_1 = 0$ $K_{t_1}(x_1)dx_1$ must be replaced by the Dirac unit mass at 0. Since the kernels K_t form a convolution semigroup $K_{t+s} = K_t * K_s,\ t > 0, s > 0$, it is easy to see that the measures P_σ form a compatible system.

We regard P_σ as the distribution of a random vector $(x(t_0), x(t_1), \ldots, x(t_n))$ with $x(t_0) = 0$ and independent increments $x(t_i) - x(t_{i-1}),\ i = 1, \ldots, n$. If A is a Borel subset of $\mathbb{R}^\sigma$ we denote $P_\sigma(A)$ by a notation such as

$$P_\sigma\{(x(t_0), \ldots, x(t_n)) \in A\}.$$

Wiener measure is the unique Radon measure P on the Banach space $C_0[0,T]$ such that for all finite $\sigma \subset [0,T]$

$$\pi_\sigma(P) = P_\sigma, \quad \sigma \in \mathfrak{S}.$$

We shall show, by applying Theorem 4.1, that Wiener measure, the projective limit of the measures P_σ , exists on $C_0[0,T]$.

If $0 < s < t \le 1$ and $\{s,t\} \subset \sigma$ we have, if $A \subset \mathbb{R}^2$

$$P_\sigma((x(s), x(t)) \in A) = \iint\limits_A K_{t-s}(x_2 - x_1) K_s(x_1) dx_2 dx_1.$$

In particular,

$$P_\sigma(|x(t)-x(s)|>r) = \iint\limits_{|x_2-x_1|>r} K_{t-s}(x_2-x_1)K_s(x_1)dx_2dx_1$$
$$= \int\limits_{|y_2|>r} K_{t-s}(y_2)K_s(y_1)dy_2dy_1 = \int\limits_{|y|>r} K_{t-s}(y)dy$$

so that we have, for $0 \le s < t$

$$\begin{aligned} P_\sigma(|x(t)-x(s)|>r) &= \frac{1}{\sqrt{2\pi(t-s)}}\int_{|y|>r} e^{-y^2/2(t-s)}dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{|x|>r/\sqrt{t-s}} e^{-x^2/2}dx = \varrho(\frac{t-s}{r^2}) \end{aligned} \tag{5.4}$$

where

$$\varrho(\delta) = \frac{1}{\sqrt{2\pi}}\int_{|x|>1/\sqrt{\delta}} e^{-x^2/2}dx. \tag{5.5}$$

For $p \ge 1$, $\varrho(\delta) \le \frac{1}{\sqrt{2\pi}}\int(\delta x^2)^p e^{-x^2/2}dx = C\delta^p$, in particular

$$\varrho(\delta) = o(\delta). \tag{5.6}$$

If $0 \le a < b$ let

$$\omega_{a,b}(x,\delta) = \sup_{\substack{a\le t,s\le b \\ |t-s|\le\delta}} |x(t)-x(s)|. \tag{5.7}$$

Then x is continuous on $[a,b]$ iff $\lim\limits_{\delta\to 0}\omega_{a,b}(\delta,x)=0$. Similarly we define

$$\omega_\sigma(x,\delta) = \sup_{\substack{t,s\in\sigma \\ |t-s|\le\delta}} |x(t)-x(s)|. \tag{5.8}$$

Lemma 5.1. *Let $(x_0,x_1,\dots,x_n)$ be a random vector with independent increments x_i-x_{i-1}, $i=1,\dots,n$ defined on a probability space with probability P. Then*

$$P\{\sup_{0\le k\le n}|x_k-x_0|>r\} \le 2\sup_{0\le k\le n} P\left\{|x_n-x_k|>\frac{r}{2}\right\}. \tag{5.9}$$

Proof. Let M be the supremum on the right-hand side. The sets

$$A_k = \{x : |x_i-x_0|\le r, 1\le i<k, |x_k-x_0|>r\}$$

being pairwise disjoint and having union equal to

$$\{x : \sup_{0\le k\le n}|x_k-x_0|>r\}$$

we have

$$P\{\sup_{0\le k\le n}|x_k-x_0|>r\}=\sum_{k=1}^{n}P\{|x_i-x_0|\le r,1\le i<k,|x_k-x_0|>r\}$$

$$=\sum_{k=1}^{n}P\left\{|x_i-x_0|\le r,1\le i<k,|x_k-x_0|>r,|x_n-x_k|\le\frac{r}{2}\right\}$$

$$+\sum_{k=1}^{n}P\left\{|x_i-x_0|\le r,1\le i<k,|x_k-x_0|>r,|x_n-x_k|>\frac{r}{2}\right\}.$$

The first sum is at most equal to

$$\sum_{k=1}^{n}P\left\{|x_i-x_0|\le r,1\le i<k,|x_k-x_0|>r,|x_n-x_0|>\frac{r}{2}\right\}$$

because $|x_k-x_0|>r$ and $|x_n-x_k|\le r/2$ imply $|x_n-x_0|>r/2$. The sets A_k being disjoint, this equals

$$P\left\{\sup_{0\le k\le n}|x_k-x_0|>r,|x_n-x_0|>\frac{r}{2}\right\}\le P\left\{|x_n-x_0|>\frac{r}{2}\right\}\le M.$$

The second sum, by the independence of the increments, equals

$$\sum_{k=1}^{n}P\{|x_i-x_0|\le r,1\le i<k,|x_k-x_0|>r\}P\left\{|x_n-x_k|>\frac{r}{2}\right\}$$

$$=P\{\sup_{0\le k\le n}|x_k-x_0|>r\}P\left\{|x_n-x_k|>\frac{r}{2}\right\}\le M.$$

Together this gives the required estimate. □

We apply this with $(x_0,x_1,\dots,x_n)=(x(t_0),x(t_1),\dots,x(t_n))$ and $P=P_\sigma$.

Lemma 5.2. *Let $\sigma=\{t_0,\dots t_n\}$ be a subdivision of $[a,b]$ with $a=t_0<t_1<\dots<t_n=b$. Then*

$$P_\sigma\{\sup_{t,s\in\sigma}|x(t)-x(s)|>4r\}\le 4\varrho((b-a)/r^2). \tag{5.10}$$

Proof. $|x(t)-x(s)|>4r$ implies $|x(t)-x(a)|>2r$ or $|x(s)-x(a)|>2r$. Therefore

$$P_\sigma\{\sup_{t,s\in\sigma}|x(t)-x(s)|>4r\}\le 2P_\sigma\{\sup_{t\in\sigma}|x(t)-x(a)|>2r\}.$$

By Lemma 5.1 this at most equal to

$$4\sup_{t\in\sigma}P_\sigma(|x(b)-x(t)|>r)\le 4\varrho\left(\frac{b-a}{r^2}\right).$$ □

Lemma 5.3. *Let σ be a subdivision of $[a,b]$, and let $\delta\le(b-a)/2$. Then*

$$P_\sigma\{\omega_\sigma(x,\delta)>4r\}\le\frac{2(b-a)}{\delta}\varrho(4\delta/r^2). \tag{5.11}$$

Proof. Let n be the smallest integer so that $n \geq (b-a)/2\delta$. Then

$$\frac{b-a}{n} \leq 2\delta < \frac{b-a}{n-1}. \tag{5.12}$$

Let $a = a_0 < \cdots < a_n = b$ be the subdivision into equal intervals such that $a_i - a_{i-1} = \frac{b-a}{n}$. Then, δ being smaller than the length of these intervals, it follows that if $|t-s| \leq \delta$, t and s either belong to one of these intervals or to the union of two consecutive ones. Therefore $\sup\limits_{\substack{t,s\in\sigma \\ |t-s|\leq\delta}} |x(t)-x(s)| > r$ implies that there exists at least one set $[a_{i-1}, a_{i+1}] := \{t \in \sigma : a_{i-1} \leq t \leq a_{i+1}\}$, with $1 \leq i \leq n-1$, such that $\sup\limits_{t,s\in[a_{i-1},a_{i+1}]} |x(t)-x(s)| > r$. Hence, putting $T = b-a$, we have by Lemma 5.2

$$\begin{aligned} P_\sigma\{\omega_\sigma(x,\delta) > 4r\} &\leq \sum_{i=1}^{n-1} P_\sigma\big\{\sup_{t,s\in[a_{i-1},a_{i+1}]} |x(t)-x(s)| > 4r\big\} \\ &\leq 4(n-1)\varrho(\frac{2T}{nr^2}) \leq \frac{2T}{\delta}\varrho(\frac{4\delta}{r^2}). \end{aligned}$$

□

Now, by (5.6), and Lemma 5.3, we have for any $r > 0$

$$\lim_{\delta\searrow 0} P_\sigma\{\omega_\sigma(x,\delta) > r\} = 0 \tag{5.13}$$

uniformly with respect to the subdivisions σ of $[a,b]$.

Let δ_n be a sequence of positive numbers tending to zero, such that for all subdivisions σ of $[0,T]$ we have

$$P_\sigma\{\omega_\sigma(x,\delta_n) > \frac{1}{n}\} \leq \frac{1}{n^2}. \tag{5.14}$$

Let N be such that $\sum_{n\geq N} \frac{1}{n^2} \leq \varepsilon$.

Let

$$\begin{aligned} K_\sigma &= \left\{x \in \mathbb{R}^\sigma : x(0) = 0, \omega_\sigma(x,\delta_n) \leq \frac{1}{n} \text{ for all } n \geq N\right\} \\ &= \left\{x \in \mathbb{R}^\sigma : x(0) = 0,\ t,s \in \sigma, n \geq N, |t-s| \leq \delta_n \implies |x(t)-x(s)| \leq \frac{1}{n}\right\}. \end{aligned}$$

The projective limit of the spaces $\mathbb{R}^\sigma$ is just $\mathbb{R}^{[0,T]}$. Clearly we have, if $\sigma \leq \sigma'$,

$$\pi_{\sigma\sigma'}(K_{\sigma'}) \subset K_\sigma.$$

The projective limit of the sets K_σ is

$$\begin{aligned} K = \Big\{x \in \mathbb{R}^{[0,T]} &: x(0) = 0,\ t,s \in [0,T],\ n \geq N,\ |t-s| \leq \delta_n \\ &\implies |x(t)-x(s)| \leq \frac{1}{n}\Big\} \\ = \Big\{x \in \mathbb{R}^{[0,T]} &: x(0) = 0, \omega(x,\delta_n) \leq \frac{1}{n} \text{ for all } n \geq N\Big\}. \end{aligned}$$

By Ascoli's theorem this is a compact subset of $C_0[0,T]$. Moreover

$$P_\sigma(K_\sigma^c) \leq \sum_{n\geq N} P_\sigma\{x \in \mathbb{R}^\sigma : \omega_\sigma(x,\delta_n) > \frac{1}{n}\} \leq \sum_{n \geq N} \frac{1}{n^2} \leq \varepsilon.$$

By Theorem 2 Wiener measure, the projective limit of the measures P_σ, exists on $C_0[0,T]$.

Remark 5.4. Let $\mathrm{mod}(\sigma) = \min\{|t-s| : t,s \in \sigma, t \neq s\}$. If $\delta_n < \mathrm{mod}(\sigma)$, then $\omega_\sigma(x,\delta_n) = 0$ and the corresponding condition in the definition of K_σ is void. If $\mathrm{mod}(\sigma) \leq \delta_1$ at least one of the conditions is not void, and K_σ is compact.

Remark 5.5. While it is easy to see that $\pi_\sigma(K) \subset K_\sigma$, a description of $\pi_\sigma(K)$ seems not to be practicable, depending as it does on the distances between the points of σ and on the values of δ_n.

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The H^∞ Holomorphic Functional Calculus for Sectorial Operators – a Survey

Lutz Weis

It is a pleasure to dedicate this article to Philippe Clément in deep appreciation of his path breaking mathematical contributions and his enthusiastic personality radiating the joy of mathematics.

Abstract. In this article we survey recent results on the holomorphic functional calculus for sectorial operators on Banach spaces. Starting from important classes of operators with a bounded H^∞-calculus and its essential applications we show how the classical Hilbert space theory of the H^∞-calculus can be extended to the Banach space setting. In particular, we discuss characterizations in terms of dilations, interpolation theory and square function estimates. These results lead to perturbation results which in turn allow us to verify the boundedness of the H^∞-calculus for new classes of partial differential operators.

1. Introduction

The H^∞-functional calculus is an extension of the classical Dunford calculus for bounded operators to unbounded sectorial operators, which was formalized by A. McIntosh and his collaborators in [18], [70]. They were mainly motivated by the connection of the H^∞-calculus with Kato's square root problem, and indeed the H^∞-calculus made a contribution to the recent solution of this longstanding problem. A second source of inspiration for the H^∞-calculus was the operator-method approach to evolution equations of Grisvard and Da Prato, in particular, the characterization of maximal regularity and the identification of fractional powers of differential operators. These important applications have motivated much work on the H^∞-calculus for partial differential operators. By now we know that large classes of elliptic differential operators with rather general coefficients and boundary values, Schrödinger operators with singular potentials and many Stokes operators have a bounded H^∞-calculus. This large pool of examples makes a stark contrast to earlier attempts to develop a spectral theory on Banach spaces, more closely modelled after the theory for normal operators on Hilbert spaces.

The diversity of applications and examples led to more general questions about the H^∞-calculus. Can we construct a joint functional calculus for (non-) commuting operators? Are there general criteria for the boundedness of the H^∞-calculus, e.g., practical perturbation theorems? What is the connection of the H^∞-calculus with the interpolation of domains? These questions were first studied for Hilbert space operators in connection with dilations, imaginary powers and square function estimates (see, e.g., [70], [72], [65], [9]). More recently a theory of the H^∞ calculus on Banach spaces is emerging, which extends many of the essential Hilbert spaces results to the Banach space setting (e.g., to $L_p(\Omega)$-spaces). This requires methods from Banach space theory, in order to extend the harmonic analysis techniques of McIntosh. In this report we mainly survey the latter development (see, e.g., [41], [54], [55], [56], [75]). However, we first give a brief review of some of the applications of the H^∞-calculus and the main classes of operators for which the H^∞-calculus is bounded.

We have concentrated on the H^∞-calculus for sectorial operators, but there is a parallel theory for bisectorial operators and operators of strip type, to which we refer to occasionally.

This report is subdivided into the following sections:

1. Introduction
2. Construction of the H^∞-functional calculus
3. Examples for operators with a bounded H^∞-functional calculus
4. Some application of the H^∞-functional calculus
5. H^∞-functional calculus for Hilbert space operators
6. A randomization technique for the H^∞-functional calculus
7. Characterization by dilations and imaginary powers
8. Perturbation theorem for the H^∞-calculus
9. H^∞-functional calculus and interpolation
10. Joint functional calculus and operator sums

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2. Construction of the H^∞-functional calculus

A convenient class for the construction of a functional calculus are the sectorial operators. A linear operator on a Banach space X with dense domain $\mathcal{D}(A)$ and dense range $\mathcal{R}(A)$ is called sectorial if its spectrum $\sigma(A)$ is contained in a sector

$$\Sigma_\omega = \{\lambda \in \mathbb{C} : \ |\arg \psi| < \omega\} \cup \{0\}. \tag{2.1}$$

and there is a constant $C_\omega < \infty$ so that

$$\|\lambda R(\lambda, A)\| \le C_\omega \text{ for } \lambda \notin \Sigma_\omega.$$

The infimum over all such ω is denoted by $\omega(A)$.

We would like to construct a functional calculus of holomorphic functions on Σ_σ with $\sigma > \omega$ so that we can define operators such as e^{tA} (if $\sigma < \frac{\pi}{2}$, $t \in \mathbb{R}_+$), A^λ with $\lambda \in \mathbb{C}$ or $\lambda^{2-\alpha} A^\alpha R(\lambda, A)^2$ with $\alpha \in (0, 2)$, study their algebraic relationships and give norm estimates if they are bounded. This will be done in two steps, following the approach of A. McIntosh.

Step 1: First we consider special holomorphic functions $f : \Sigma_\sigma \to \mathbb{C}$ which satisfy an estimate

$$|f(\lambda)| \leq C \Big| \frac{\lambda}{(1+\lambda)^2} \Big|^\varepsilon, \ \lambda \in \Sigma_\sigma$$

for some $\varepsilon > 0$. We denote this class by $H_0^\infty(\Sigma_\sigma)$. If we assume for a moment, that A belongs to $B(X)$, the space of bounded linear operators on X, with $\sigma(A) \subset \Sigma_\omega \cap \{\lambda : \frac{1}{n} < |\lambda| < n\}$ for some $\omega \in (0, \pi)$ and $n \in \mathbb{N}$, then we can apply the classical Dunford calculus for bounded operators and define for every $f \in H_0^\infty(\Sigma)$ with $\sigma > \omega(A)$ the bounded operator

$$f(A) = \frac{1}{2\pi i} \int_{\partial G_n} f(\lambda) R(\lambda, A) d\lambda$$

where $G_n = \Sigma_\nu \cap \{\lambda : \frac{1}{n} < |\lambda| < n\}$ with $\omega(A) < \nu < \sigma$.

By Cauchy's theorem we obtain for $n \to \infty$

$$f(A) = \frac{1}{2\pi i} \int_{\Sigma_\nu} f(\lambda) R(\lambda, A) d\lambda. \tag{2.2}$$

Note, that this integral makes sense as a Bochner integral for all sectorial operators (not just for bounded ones) and we can use (2.2) to define $f(A)$ for sectorial operators A and $f \in H_0^\infty$.

Step 2: In a second step we consider polynomially bounded functions. For $\alpha \geq 0$ we denote by $H_\alpha(\Sigma_\sigma)$ the space of holomorphic functions $f : \Sigma_\sigma \to \mathbb{C}$ for which $|\lambda^\alpha f(\lambda)|$ is bounded for $|\lambda| \leq 1$ and $|\lambda^{-\alpha} f(\lambda)|$ is bounded for $|\lambda| \geq 1$. This is equivalent to

$$\|f\|_{H_\alpha(\Sigma_\sigma)} \ = \ \sup\{|\rho(\lambda)|^\alpha |f(\lambda)| : \lambda \in \Sigma_\sigma\} < \infty$$

where $\rho(\lambda) = \lambda(1+\lambda)^{-2}$. Since $\rho \in H_0^\infty(\Sigma_\sigma)$ we can define $\rho(A)$ by (2.2) and check that $\rho(A)^{-1} = 2Id + A + A^{-1}$, and that $\mathcal{D}(\rho(A)^{-k}) = \mathcal{R}(\rho(A)^k) = \mathcal{D}(A^k) \cap \mathcal{R}(A^k)$, $k \in \mathbb{N}$, are dense subspaces of X. To define $f(A)$ for $f \in H_\alpha(\Sigma_\sigma)$ we choose a $k \in \mathbb{N}$ with $k > \alpha$ and write

$$f(\lambda) = \rho(\lambda)^{-k} (\rho^k f)(\lambda)$$

the point is that $\rho^k f$ also belongs to $H_0^\infty(\Sigma_\sigma)$ and we can form $(\rho^k f)(A)$ according to (2.2).

This motivates the following definition of $f(A)$.

$$\mathcal{D}(f(A)) = \{x \in X, (\rho^k f)(A)x \in \mathcal{D}(\rho(A)^{-k})\}$$
$$f(A) = \rho(A)^{-k} (\rho^k f)(A).$$

One can show that $f(A)$ is a well-defined closed operator and that the usual properties of a holomorphic functional calculus hold. In particular we have the *algebraic* properties

- If $f, g \in H_\alpha(\Sigma_\sigma)$ then $\mathcal{D}(f(A)) \cap \mathcal{D}(g(A)) \subset \mathcal{D}((f+g)(A))$ and $(f+g)(A)x = f(A)x + g(A)x,\ x \in \mathcal{D}(f(A)) \cap \mathcal{D}(g(A))$
- If $f \in H_\alpha(\Sigma_\sigma), g \in H_\beta(\Sigma_\sigma)$ then $\mathcal{D}(f(A)g(A)) = \mathcal{D}((f \cdot g)(A)) \cap \mathcal{D}(g(A))$ and $(f \cdot g)(A)x = f(A)[g(A)x]$ for $x \in \mathcal{D}(f(A)g(A))$

and the important 'convergence' property

- If $f_n \in H_\alpha(\Sigma_\sigma)$ with $\|f_n\|_{H_\alpha(\Sigma_\sigma)} \le C$ and $f_n(\lambda) \to f(\lambda)$ for $\lambda \in \Sigma_\sigma$, then for all $x \in \mathcal{D}(A^k) \cap \mathcal{R}(A^k)$ with $k > \alpha$
$$\lim_{n\to\infty} f_n(A)x = f(A)x.$$

With this calculus one defines for a sectorial operator fractional powers $A^\lambda, \lambda \in \mathbb{C}$ simply as $A^z = f_z(A)$, where $f_z(\lambda) = \lambda^z = e^{z \log \lambda}$. We use the branch of the logarithm analytic in $\mathbb{C} \setminus \mathbb{R}_-$. Now one can prove in an efficient way most of their properties presented in [58], including

- $A^{z_1+z_2}x = A^{z_1}A^{z_2}x \quad$ for $x \in \mathcal{D}(A^k) \cap \mathcal{R}(A^k),\ k > |z_1| + |z_2|$
- $(A^\alpha)^z = A^{\alpha z}$ if $\alpha \in \mathbb{R},\ |\alpha| < \frac{\pi}{\omega(A)}$

and representation formulas such as

- $A^z x = \frac{\sin(\pi z)}{\pi} \int_0^\infty t^{z-1}(t+A)^{-1}Ax\, dt, \quad x \in \mathcal{D}(A), \quad 0 < \operatorname{Re} z < 1.$

Traditionally such formulas are used as the definition of A^z and the starting point of the theory of fractional powers.

Step 3: A functional calculus is an important tool to obtain norm estimates for operators $f(A)$. Of course, for $f \in H_0^\infty(\Sigma_\sigma)$ we obtain from (2.2) the trivial estimate
$$\|f(A)\| \le C \int_{\partial\Sigma_\nu} |f(\lambda)| \frac{|d\lambda|}{|\lambda|}$$
where $C = \frac{1}{2\pi} \sup_{\lambda \in \Sigma\nu} \|\lambda R(\lambda, A)\|$. But if f belongs merely to the space $H^\infty(\Sigma_\sigma)$ of bounded holomorphic functions on Σ_σ then, since $\|R(\lambda, A)\| \approx \frac{1}{|\lambda|}$ on $\partial\Sigma_\nu$, (2.2) turns into a singular integral which may or may not exist. A good example is provided by the functions $f_t = \lambda^{it}$, $t \in \mathbb{R}$, which belong to $H^\infty(\Sigma_\sigma)$, but it is well known that the imaginary powers $A^{it} = f_t(A)$ of a sectorial operator are not always bounded (cf. [58]). Therefore we say that A has a *bounded H^∞-functional calculus* for $\sigma > \omega(A)$, if there is a constant C such that
$$\|f(A)\|_{B(X)} \le C\|f\|_{H^\infty(\Sigma_\sigma)} = C \sup_{z \in \Sigma_\sigma} |f(z)|. \tag{2.3}$$
In this case one can extend the map $f \in H_0^\infty(\Sigma_\sigma) \to F(A) \in B(X)$ defined by (2.2) to a bounded algebra homomorphism
$$f \in H^\infty(\Sigma_\sigma) \to f(A) \in B(X).$$
By $\omega_{H^\infty}(A)$ we denote the infimum over all angles σ for which (2.3) holds.

We will see that in many situations sharp estimates require such a bounded H^∞-calculus.

Although there are sectorial operators without a bounded H^∞-calculus even in Hilbert space (cf. [72]), we will exhibit in Sections 3 and 7 large classes which are of operators of interest in applications and do have a bounded H^∞-calculus, e.g., elliptic differential operators, Schrödinger operators and Stokes operators.

Note. This construction appeared first in [70] and [2]. For details and further information we refer to [2], [20], [25], [61, Sections 9 and 15], [45], [48], [79]. In these references it is also explained how to handle operators that are not injective. For spectral mapping theorems for this calculus see [46].

A similar holomorphic functional calculus can be constructed for bisectorial operators (replace Σ_σ by $\Sigma_\sigma \cup (-\Sigma_\sigma)$ in the definition of sectoriality (see [9], [2], [28], [29])) and for operators A of strip type, i.e., $\sigma(A)$ is contained in a strip $S_a = \{\lambda : |Re\lambda| < a\}$ and the spectrum is bounded outside of S_a. A good example of such an operator is $i \log A$ for a sectorial operator A (see [13], [14], [47], [57]).

3. Examples for operators with a bounded H^∞-functional calculus

For a sectorial operator A on Hilbert space, which is selfadjoint or normal, the H^∞-functional calculus is of course just a special case of the more powerful functional calculus for bounded Borel functions. But if one considers $A = -\Delta$ on $L_p(\mathbb{R}^n)$ for $p \neq 2$ or $A = a\Delta$ with variable coefficients $a \in L_\infty(\mathbb{R}^n)$ on $L_2(\mathbb{R}^n)$, purely spectral theoretic methods do not seem to be sufficient any more. However for many examples ideas from harmonic analysis will help to bound the *singular* integral

$$f(A) = \int_{\partial\Sigma_\nu} f(\lambda)R(\lambda, A)d\lambda, \;\; f \in H^\infty(\Sigma_\sigma).$$

3.1. The Laplacian

Formally, $A = -\Delta$ is given on $X = L_p(\mathbb{R}^n)$ as a Fourier multiplier operator

$$-\Delta = \mathcal{F}^{-1} M_m \mathcal{F}$$

where $\mathcal{F}$ denotes the Fourier transform and M_m is the pointwise multiplication operator with $m(u) = |u|^2$.

For $f \in H_0^\infty(\Sigma_\sigma)$, $\sigma > 0$, we have then $f(M_m) = M_{f \circ m}$ and $f(-\Delta)$ corresponds to the Fourier multiplier operator

$$f(-\Delta) = \mathcal{F}^{-1} M_{f \circ m} \mathcal{F}.$$

The boundedness of this operator on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$ can be established, e.g., with the help of Marcinkiewicz's multiplier theorem. For this we have to check that for all multiindices $\alpha \leq (1, \ldots, 1)$ of order m and $u \in \mathbb{R}^n$

$$|u^\alpha| \left|D^\alpha[f(|u|^2)]\right| \leq C \, |u|^{2m} \left|f^{(m)}(|u|^2)\right|$$

is uniformly bounded on $\mathbb{R}^n$. Indeed, since $f \in H^\infty(\Sigma_\sigma)$ we get from Cauchy's integral formula that

$$\sup_{t>0} \left| t^m f^{(m)}(t) \right| < \infty$$

and the claim follows.

If one considers $-\Delta$ on $L_p(\mathbb{R}^n, X)$, then $-\Delta$ has a bounded H^∞-calculus on this space if and only if $1 < p < \infty$ and X is a UMD-space (see, e.g., [50], [61]).

3.2. Operators of diffusion type

Using more advanced methods from harmonic analysis (e.g., square function estimates) Stein showed (implicitly in Section IV.6 of [78]) that operators A such that

a.) A is selfadjoint on $L_2(\mathbb{R}^n)$
b.) $-A$ generates a positive contractive semigroup T_t on $L_q(\mathbb{R}^n)$, $1 \le q \le \infty$
c.) $T_t 1 = 1$, $T_t^* 1 = 1$

have a bounded H^∞-calculus on all $L_p(\mathbb{R}^n)$, $1 < p < \infty$, with $\omega_{H^\infty}(A) < \pi$. Cowling ([17]) gave a more direct proof of this fact, using the Coifman-Weiss transference and eliminated assumption (iii). Duong ([33]) and Hieber and Prüss ([50]) made a big step forward by showing that A has a bounded H^∞-calculus if $-A$ generates a positive contractive semigroup T_t on a general $L_p(\Omega, \mu)$-space for *one* $p \in (1, \infty)$ at least for $\sigma > \pi/2$. Finally it was shown in [55], [80] (see also [61]) that, if in addition T_t extends to an analytic semigroup bounded on a sector Σ_ω then $\omega_{H^\infty}(A) < \pi/2$. These results certainly cover all generators of stochastic processes.

The H^∞-calculus for 'diffusions' on non-commutative L_p-spaces was studied recently in [52].

3.3. Operators with kernel bounds

Assume that A has a bounded H^∞-calculus on $L_2(\Omega, \mu)$ where (Ω, d, μ) is $\mathbb{R}^n$ or, more generally, a space of homogeneous type of dimension n with metric d. If T_t is an integral operator with a kernel $k_t(u, v)$ satisfying a Gaussian estimate

$$|k_t(u,v)| \le C(2\pi t)^{n/2} e^{-|u-v|^2/(bt)}, \quad u, v \in \Omega \tag{3.1}$$

with constants C and b, or if the resolvent $R(\lambda, A)$ satisfies more general *Poisson* estimates, then it is possible to *extrapolate* A to an operator on $L_p(\Omega)$ with a bounded H^∞-calculus for all $1 < p < \infty$. See [36] and [35], where variants of Hörmander's condition and the basic estimate of Calderon-Zygmund theory is used. This approach was considerably improved in [12] by replacing the kernel estimates (3.1) by *weighted norm estimates.* For fixed $1 \le p < 2 < q \le \infty$ assume that

$$\left\| \chi_{B(u,t^{1/m})} e^{-tA} \chi_{B(v,t^{1/m})} \right\|_{L_p \to L_q} \le \mu(B(x, t^{1/m})^{-(1/p-1/q)} g\Big(\frac{d(u,v)}{t^{1/m}}\Big) \tag{3.2}$$

for all $t > 0$, $u, v \in \Omega$, where $g : [0, \infty) \to [0, \infty)$ is a non-increasing function that decays faster than any polynomial. Then (3.2) implies that A induces consistent operators A_s on $L_s(\Omega,\mu)$ for all $s\in(p,q)$ with $\omega_{H^\infty}(A_s)=\omega_{H^\infty}(A)=\omega(A)$. This result was the first in this area which could treat operators generating a semigroup only on part of the L_p-scale.

3.4. Elliptic partial differential operators

The results of 3.2 and 3.3 cover already a large class of elliptic differential operators, but there is also a more direct approach combining pseudodifferential techniques and perturbation theorems. Consider systems of the form

$$(Ax)(u)=\sum_{|\alpha|\le 2m} a_\alpha(D^\alpha x)(u), \qquad u\in\Omega\subset\mathbb{R}^n \tag{3.3}$$

where $D^\alpha=(-i\frac{\partial}{\partial x_1})^{\alpha_1}\cdots(-i\frac{\partial}{\partial x_n})^{\alpha_n}$ and the $\mathbb{C}^{N\times N}$-valued symbol $a(u,v)=\sum_{|\alpha|=2m}a_\alpha(u)v^\alpha$ is elliptic in the sense that for some $\omega\in[0,\infty)$ and all $u\in\mathbb{R}^n$, $|v|=1$

$$a_\alpha\in L_\infty(\Omega),\quad \sigma(a(u,v))\subset\overline{\Sigma_\omega}\setminus\{0\},\quad \left\|a(u,v)^{-1}\right\|\le M.$$

For $\Omega=\mathbb{R}^n$ and Hölder continuous coefficients proofs for the boundedness of the H^∞-calculus on $L_p(\Omega,\mathbb{C}^n)$ of A were given in [4], [20], [54], for continuous coefficients in [31] and for VMO-coefficients in [34] and [49]. For the most general results on a region Ω with Hölder continuous coefficients and Lopatinskij-Shapiro boundary conditions see [21] and [54] for a different approach.

For operators in divergence form, i.e., operators determined by a form

$$a(x,y)=\int_{\mathbb{R}^n}\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(\partial^\alpha x)(\partial^\beta y)du \tag{3.4}$$

where $x,y\in H_2^m(\mathbb{R}^n)$, satisfying a sectoriality and an ellipticity condition

$$|\operatorname{Im} a(x,x)|\le\tan\psi\operatorname{Re} a(x,x),\quad \operatorname{Re} a(x,x)\ge\delta\left\|(-\Delta)^{m/2}x\right\|_2^2$$

for $x\in H_2^m$ and some $\psi\in[0,\pi/2]$ and $\delta>0$ we can apply 3.3 to obtain operators on $L_p(\mathbb{R}^n)$ with a bounded H^∞-calculus (cf. [32], [36], [35], [54]). We will come back to elliptic operators in Sections 8 and 9.

3.5. Stokes operators

For a bounded domain $\Omega\subset\mathbb{R}^n$ with a $C^{1,1}$-boundary $\partial\Omega$ we denote by P_p the Helmholtz projection of $L_p(\Omega)^n$ onto the Helmholtz space $L_{p,\sigma}(\Omega)^n$ for $1<p<\infty$. The Stokes operator has then the form $A_p=P_p\Delta$ with Dirichlet boundary values. The boundedness of the H^∞-calculus of A_p is shown in [73] and [54] and based on maximal regularity results in [40]. For earlier results on BIP for Stokes operators see [42].

3.6. L_1 and $C(K)$-spaces

In the previous example we always encountered the restriction $1 < p < \infty$. This is no accident. In $L_1(\Omega, \mu)$ and $C(K)$ there are no interesting examples of unbounded operators with a bounded H^∞-calculus, certainly no differential operators. E.g. the Gaussian bound (3.1) which often guarantees the boundedness of the H^∞-calculus for L_p, $1 < p < \infty$, excludes it in the L_1-case. See [55], [51] and [59] for this and more results in the same direction. This situation reflects the well-known fact that singular integral operators are unbounded on $L_1(\mathbb{R}^n)$ and $C_b(\mathbb{R}^n)$ spaces.

3.7. Besov spaces

If A is any sectorial operator on a Banach space X, then A induces sectorial operators $A_{\theta,q}$ on all real interpolation spaces $X_{\theta,q} = (X, \mathcal{D}(A))_{\theta,q}$ which have always a bounded H^∞-functional calculus (cf. [23], [24]). If $X = L_p(\mathbb{R}^n)$ and $\mathcal{D}(A) = W^{2,p}(\mathbb{R}^n)$, then $X_{\theta,q}$ are the Besov spaces B^s_{pq} with $1 \leq p, q \leq \infty$ (see [11]) and it follows that all differential operators considered in 3.4 induce operators on $B^s_{pq}(\mathbb{R}^n)$ with a bounded H^∞-calculus if only they are sectorial on L_p no matter whether they have a bounded H^∞-calculus on $L_p(\mathbb{R}^n)$. In contrast to 3.6 the cases $p = 1$ and $p = \infty$ are not excluded.

4. Some applications of the H^∞-functional calculus

We mentioned already in the context of fractional powers that the holomorphic functional calculus is quite effective in reducing calculation with operator functions to the corresponding calculations with holomorphic functions. By providing norm estimates for operators $f(A)$ it is also quite useful in establishing sharp estimates in various analytic contexts. Here are some examples.

4.1. Identification of $D(A^\alpha)$

If A has a bounded H^∞-calculus it has bounded imaginary powers, and under this assumption it was shown by Yagi ([82], [83], [61, Section 15]) that the domains of fractional powers of A can be obtained by complex interpolation

$$\mathcal{D}(A^\alpha) = [X, \mathcal{D}(A)]_\alpha, \qquad 0 < \alpha < 1.$$

For many elliptic differential operators this means that $\mathcal{D}(A^\alpha)$ can be identified with a Sobolev space. For $A = -\Delta$ on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, we have, e.g., $\mathcal{D}(A^\alpha) = W^{2\alpha}_p(\mathbb{R}^n)$. Early results for more general differential operators are due to Seeley [76]. Hence we obtain a useful connection between spectral theory and analysis, which allows, e.g., to *measure* regularity of initial values or solutions of differential equations.

4.2. Spectral projections

If we replace the sector Σ_ω by bisectors $\Sigma^b_\omega = \Sigma_\omega \cup (-\Sigma_\omega)$ with $\omega \in (0, \pi/2)$, then we obtain the definition of *bisectorial operators*. It was pointed out in [9], that for such operators one can construct a functional calculus for functions holomorphic on Σ_σ^b for $\sigma > \omega$ along the lines of Section 2.

If A is a bisectorial operator with a bounded $H^\infty(\Sigma_\omega^b)$-functional calculus we can define bounded *spectral projections* P_+, P_- with respect to the *spectral sets* $\sigma(A) \cap \Sigma_\omega$ and $\sigma(A) \cap (-\Sigma_\omega)$, simply by

$$P_\pm = p_\pm(A), \qquad p_+ = \chi_{\Sigma_\omega}, \quad p_- = \chi_{\Sigma_\omega^b} - p_+.$$

As a consequence of the boundedness of P_+ one can derive an important estimate for the sectorial operator A^2 (with $\omega(A^2) \le 2\omega$):

$$\|Ax\| \approx \left\|(A^2)^{1/2}x\right\| \qquad \text{for } x \in D(A) = D((A^2)^{1/2}). \tag{4.1}$$

This estimate is even equivalent to the boundedness of the spectral projection P_+ as shown in [27] (see also [29]).

Example 4.1. Consider the operator $A = i\frac{d}{dz}$ on $L_p(\mathbb{R}, X)$. Then A is bisectorial since iA generates the translation group on $L_p(\mathbb{R}, X)$. Since A is a Fourier multiplier with respect to the function $m(t) = t$ we see that P_+ has the form $P_+ = \frac{1}{2}(I + iH)$, where H is the Hilbert transform on $L_p(\mathbb{R}, X)$. Hence the classical estimate

$$\left\|\frac{d}{dt} f\right\|_{L_p(\mathbb{R},X)} \approx \left\|\Big(-\frac{d^2}{dt^2}\Big)^{1/2} f\right\|_{L_p(\mathbb{R},X)} \tag{4.2}$$

holds if and only if H is bounded on $L_p(\mathbb{R}, X)$, i.e., if and only if $1 < p < \infty$ and X is a UMD space. Of course, (4.2) also follows from the classical fact that $(-\frac{d^2}{dt^2})^{1/2} = H\frac{d}{dt}$, but (4.1) also addresses more general Riesz transforms.

4.3. Kato's square root problem

Much of the work of A. McIntosh and his collaborators was motivated by *Kato's square root problem* for forms. In applications many operators on a Hilbert space H are given by a continuous, sesquilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{C}$, which is coercive with respect to H, i.e., $V \hookrightarrow H \hookrightarrow V'$ and

$$\operatorname{Re} a(u,u) \ge c\,\|u\|_V^2\,, \qquad u \in V.$$

By the Lax-Milgram theorem $a(\cdot,\cdot)$ defines generators A and A_V of bounded analytic semigroups on H and V', respectively, such that A is the part of A_V in H. Now Kato's square root problem asks whether $\mathcal{D}(A^{1/2}) = V$. It is known since [71] that this property is equivalent to the boundedness of the H^∞-calculus of A_V on V'.

The H^∞-calculus, the related square function estimates and above all deep results from harmonic analysis on Carleson measures were instrumental in the recent solution of Kato's problem for second order differential operators on $\mathbb{R}^n$: it is shown in [6], [7], [10] that the operators $A = -\mathrm{div}(a\nabla)$ with uniformly elliptic measurable coefficients, defined by the form $a(u,v) = \int (a\nabla u)\cdot \nabla v$ satisfy $\|\sqrt{A}f\|_{L_2} \approx \|\nabla f\|_{L_2}$. For a more recent proof using the H^∞-calculus of Dirac type operators see [8].

4.4. Operator sums

For the operator theoretical approach to evolution equations, as put forth in [19], sharp estimates for sums of operators as in the following theorem are essential.

Theorem 4.2. *Let A and B be resolvent commuting operators. Then for $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$*

$$\|Ax\| + \|Bx\| \leq C\|Ax + Bx\| \tag{4.3}$$

if either

a) (Dore-Venni [26]) *A and B have bounded imaginary powers on a UMD-space with $\omega_{BIP}(A) + \omega_{BIP}(B) < \pi/2$*

or

b) (Kalton-Weis [55]) *B has a bounded H^∞-calculus and A is R-sectorial with $\omega_{H^\infty}(B) + \omega_R(A) < \pi/2$.*

The notion of R-sectoriality and ω_R will be explained in Section 5. In a Hilbert space R-sectoriality is the same as sectoriality.

Let us illustrate this statement by the maximal L_p regularity problem for the Cauchy problem

$$y'(t) + A_0 y(t) \;=\; g(t), \quad y(0) = 0 \tag{4.4}$$

where A_0 generates a bounded analytic semigroup on a Banach space X_0. Maximal L_p-regularity, $1 < p < \infty$, requires that the solution y, belongs to $W_p^1(\mathbb{R}_+, X_0)$ if $g \in L_p(\mathbb{R}_+, X_0)$ and

$$\left\|\frac{d}{dt}y\right\|_{L_p(\mathbb{R}_+ X_0)} + \|A_0 y\|_{L_p(\mathbb{R}_+,X_0)} \leq C\|g\|_{L_p(\mathbb{R}_+,X_0)}. \tag{4.5}$$

We put $X = L_p(\mathbb{R}_+, X_0)$. Let A be the extension $(Ag)(t) = A_0[g(t)]$ of A_0 to X and let B be the derivative $\frac{d}{dt}$ on $X = L_p(\mathbb{R}_+, X_0)$. Now the maximal regularity inequality (4.5) has taken the form (4.3). Hence a) gives the well-known result that in a UMD-space BIP implies maximal regularity and b) gives the hard part of the characterization in [81] of maximal regularity in terms of R-sectoriality. Indeed, if X is a UMD-space then B has a bounded H^∞-calculus with $\omega_{H^\infty}(B) = \frac{\pi}{2}$ (The proof is similar to the one in Section 3.1, see [61, Section 10]). This example is taken from [81], see also [26], [64], [3]. For sum theorems of non-commuting operators, see [75].

For a stochastic evolution equation

$$dy(t) = A_0 y(t)\, dt + dW(t) \tag{4.6}$$

with an X_0-valued Wiener process $W(t)$ the maximal regularity one can expect of the solution y of (4.6) is that $y \in L_2([0,T], \mathcal{D}(A^{1/2}))$. It is shown in [22] that this condition is essentially equivalent to the boundedness of the H^∞-calculus of A.

4.5. Rational approximation schemes

Let the rational function r be a stable approximation of order m to the exponential function, i.e.,

a.) $|r(z) - e^z| \le C\,|z|^{m+1}$ as $z \to 0$.
b.) $|r(z)| \le 1$ for $\operatorname{Re} z \le 0$.

An example of particular interest in applications is the Crank-Nicolson scheme defined by $r(\lambda) = (1+\lambda/2)(1-\lambda/2)^{-1}$, which is of order 2.

If $-A$ generates a bounded analytic semigroup and $0 \in \rho(A)$, then, in [63] the following error estimate is derived

$$\Big\| r\big(\frac{t}{n}A\big)^n x - e^{tA}x \Big\| \le C\frac{t^j}{r^j}\,\|A^j x\| \tag{4.7}$$

for $x \in \mathcal{D}(A^j)$, $t \in [0,1]$ and $1 \le j \le m$. If one assumes in addition that $-A$ has a bounded H^∞-calculus (which we already know for most operators appearing in applications according to Section 3), then one can avoid the rather technical estimates for curve integrals of operator functions used in [63], and apply the norm estimates of the H^∞-calculus to prove (4.7) in a more straightforward manner even if $0 \in \sigma(A)$. This is worked out together with some similar error estimates in [39].

4.6. Control theory

Consider a linear control problem

$$\begin{aligned} x'(t) + Ax(t) &= Bu(t), \qquad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned}$$

where $-A$ generates an analytic semigroup T_t on a Hilbert space X, $C : \mathcal{D}(A) \to Y$ is an observation operator and B a suitable control operator. If A has a bounded H^∞-calculus, then the admissibility of C, i.e., the inequality

$$\|CT(\cdot)x\| \le c\|x\|, \qquad x \in X$$

can be characterized in terms of the boundedness of the set $\{\lambda^{1/2}C(\lambda+A)^{-1} : \lambda > 0\}$ in $B(X)$, i.e., in terms of the resolvent of A (cf. [69]). A 'dual' statement holds for control operators. For extensions of such results to the Banach space setting, see [44] and [43].

5. H^∞-functional calculus for Hilbert space operators

We first discuss the theory of operators with a bounded H^∞ functional calculus in a Hilbert space, because here it is particularly elegant and well connected to traditional topics such as contractions, normal operators and fractional powers.

Not every sectorial operator on a Hilbert space has a bounded H^∞-calculus (cf. [72]). However, there is a convenient criterium for the boundedness of the H^∞-calculus going back to Kato. A sectorial operator A on a Hilbert space H with scalar product $(\cdot,\cdot)$ is called *ω-accretive* if

$$\sigma(A) \subset \overline{\Sigma_\omega}, \quad (Ax, x) \in \overline{\Sigma_\omega} \quad \text{for all } x \in \mathcal{D}(A). \tag{5.1}$$

It is well known that accretive operators are precisely the operators determined by closed sectorial sesquilinear forms (cf. [5]). In particular, elliptic operators in gradient form are accretive. Furthermore, by the Lumer-Phillips theorem an operator is accretive if and only if $-A$ generates a contraction semigroup.

Such operators are not just examples of operators with a bounded H^∞-calculus; but conversely every operator on a Hilbert space with $\omega_{H^\infty}(A) < \pi/2$ is similar to an accretive operator. This result is due to Le Merdy ([65], [66]).

Theorem 5.1. *Let A be a sectorial operator on a Hilbert space H with $\omega(A) < \pi/2$. The following conditions are equivalent:*

a.) *A has a bounded $H^\infty(\Sigma_\sigma)$ calculus for one (all) $\sigma \in (\omega(A), \pi)$.*
b.) *For one (all) $\omega \geq \omega(A)$ there is an equivalent scalar product $(\cdot,\cdot)_\omega$ on H so that A is ω-accretive with respect to $(\cdot,\cdot)_\omega$.*
c.) *For one (all) $\omega \in (\omega(A), \pi/2)$ there is an equivalent Hilbert space norm $\|\cdot\|_\omega$ such that $-A$ generates an analytic semigroup T_z with $\|T_z\|_\omega \leq 1$ for $z \in \Sigma_{\pi/2-\omega}$.*

Note, that the equivalent norm in b) and c) has to be a Hilbert space norm. Every bounded semigroup T_t is contractive with respect to the simple equivalent norm $|||x||| = \sup_{z\in\Sigma_\sigma} \|T(z)x\|$, but this norm may not be given by a scalar product.

There is a close connection with normal operators. We will see next that a sectorial operator has a bounded H^∞ calculus if and only if it has a dilation to a normal operator. If $\omega(A) < \pi/2$, this follows directly from 5.1 and the Sz.-Nagy dilation theorem for contractions, the general case was observed in [41].

Theorem 5.2. *A sectorial operator A on a Hilbert space H has a bounded H^∞-calculus if and only if for some (all) $\sigma > \omega(A)$ there is a normal operator N on a Hilbert space G and an isomorphic embedding $J : H \to G$ such that*

$$\begin{tikzcd} H \arrow[r, "J"] \arrow[d, "f(A)"'] & G \arrow[d, "f(N)"] \\ H & G \arrow[l, "J^{-1}\circ P"] \end{tikzcd} \qquad f \in H^\infty(\Sigma_\sigma)$$

commutes, where $P : G \to J(H)$ is the orthogonal projection.

If $\omega(A) < \pi/2$, we can choose N as the generator of a unitary group and $Je^{-tA} = Pe^{-tN}J$ for $t > 0$.

For many applications the following characterizations in terms of fractional powers are important. (See [70] for the first condition and [9] for the second).

Theorem 5.3. *The boundedness of the H^∞-calculus of a sectorial operator A on a Hilbert space H is equivalent to each of the following conditions*

a.) *A has bounded imaginary powers A^{it}, $t \in \mathbb{R}$, on H.*
b.) *For some (any) $-\infty < \alpha < \beta < \infty$ and $\theta \in (0,1)$ the complex interpolation method gives*

$$[\widetilde{\mathcal{D}(A^\alpha)}, \widetilde{\mathcal{D}(A^\beta)}]_\theta = \widetilde{\mathcal{D}(A^\gamma)}, \quad \gamma = (1-\theta)\alpha + \theta\beta$$

with equivalent norms. Here $\widetilde{\mathcal{D}(A^\alpha)}$ denotes the completion of $\mathcal{D}(A^\alpha)$ with respect to $|||x||| = \|A^\alpha x\|$.

Square functions. For the proof of Theorem 5.3 McIntosh and Yagi [70], [72], [82] introduced *square functions* $\|\cdot\|_\psi$ for a sectorial operator A and each $\psi \in H_0^\infty(\Sigma_\sigma)$ with $\psi \neq 0$ and $\sigma > \omega(A)$:

$$\|x\|_\psi := \Big(\int_0^\infty \|\psi(tA)\|^2 \frac{dt}{t} \Big)^{1/2}. \tag{5.2}$$

It is shown in [70] that A has a bounded H^∞-calculus if and only if for one (all) such ψ we have

$$\|x\|_H \approx \|x\|_\psi \qquad \text{for } x \in H. \tag{5.3}$$

For $\psi(\lambda) = \lambda^{1/2}(1-\lambda)^{-1}$ and a self-adjoint A, we have indeed

$$\begin{aligned}\int_0^\infty \|\psi(tA)x\|^2 \frac{dt}{t} &= \int_0^\infty \left\|A^{1/2}R(t,A)x\right\|^2 dt = \int_0^\infty (AR(t,A)^2x, x)\, dt \\ &= \lim_{a\to 0, b\to\infty} [(AR(t,A)x,x)]_a^b = (x,x) = \|x\|^2.\end{aligned}$$

Furthermore, if $H = L_2(\mathbb{R})$, $A = -(-\Delta)^{1/2}$ and $\phi(\lambda) = \lambda e^{-\lambda}$ then (5.3) reduces to a classical Paley-Littlewood g-function (see [78])

$$\Big(\int_0^\infty \left\| t \frac{d}{dt} P_t x \right\|^2 \frac{dt}{t} \Big)^{1/2} \approx \|x\|_{L_2}$$

where P_t is the Poisson semigroup generated by A.

We indicate now how (5.3) is used in the proofs of the theorems. The sufficiency of condition 5.3a) follows now from (5.3) and the estimate

$$\|f(A)x\|_\psi \le C\|x\|_\phi, \qquad x \in H$$

where the constant C only depends on A and $\psi, \phi \in H_0^\infty(\Sigma_\sigma)$, $\sigma > \omega(A)$.

The square functions (5.2) and estimate (5.3) are also instrumental in the proof of Theorems 5.1 and 5.2. Note for example that for $\phi(\lambda) = \lambda^{1/2}e^{-\lambda}$ (5.2) takes the form

$$\|x\|_\phi = \int_0^\infty \left\|A^{1/2}T_t x\right\|^2 dt$$

and it is easily seen that T_t is contractive with respect to the norm $\|\cdot\|_\phi$.

6. A randomization technique for the H^∞-functional calculus

In this section we outline an extension of the square function method of Section 5 to the Banach space setting. With this tool one can generalize many of the Hilbert space results of Section 5 to large classes of Banach spaces. This will be explained in Sections 7 and 9.

We have seen in Section 5 that a square function estimate

$$\|x\| \approx \Big(\int_0^\infty \|\psi(tA)x\|^2 \frac{dt}{t}\Big)^{\frac{1}{2}}, \ \psi \in H_0^\infty \tag{6.1}$$

characterizes the boundedness of the H^∞-functional calculus of a sectorial operator on a Hilbert space X. This is not true for other Banach spaces such as $L_q(\Omega)$, $1 < q < \infty$, even if we replace the exponent 2 by q. To find a good 'replacement' for (6.1) we reformulate (6.1), X still being a Hilbert space

$$\begin{aligned}\int_0^\infty \|\psi(tA)x\|^2 \frac{dt}{t} &= \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \|\psi(tA)x\|^2 \frac{dt}{t}\\ &= \int_1^2 \sum_k \|\psi(2^k tA)x\|^2 \frac{dt}{t} = \int_1^2 \Big\|\sum_{k\in\mathbb{Z}} \varepsilon_k \psi(2^k tA)x\Big\|^2_{L_2(\Omega,X)} \frac{dt}{t}\end{aligned}$$

where ε_k is a sequence of independent symmetric, $\{-1,1\}$-valued random variables on Ω (e.g., formed by a Rademacher sequence $r_n(t) = \operatorname{sign}[\sin(2^n\pi t)]$, $n \in \mathbb{N}$, on $[0,1]$). We also used the fact that $t \to \varepsilon_k\psi(2^k tA)x$ is an orthogonal sequence in the Hilbert space $L_2(\Omega, X)$. So our candidates for a *square function* in general Banach spaces X will be

$$\|x\|_\psi = \Big(\int_\Omega \Big\|\sum_{k\in\mathbb{Z}} \varepsilon_k(\omega)\psi(2^k A)x\Big\|^2 d\omega\Big)^{1/2} \approx \int_\Omega \Big\|\sum_{k\in\mathbb{Z}} \varepsilon_k(\omega)\psi(2^k A)x\Big\| \, d\omega.$$

The equivalence of the two norms follows from Kahane's inequality (e.g., [61, Section 2]). To estimate such random series we need a related 'boundedness' notion. We say that a set of operators $\tau \subset B(X)$ is *R-bounded* if there is a constant C, such that for every choice of $T_1, \dots, T_n \in \tau, x_1, \dots, x_n \in X$ and $n \in \mathbb{N}$

$$\Big\|\sum_{j=1}^n \varepsilon_j T_j x_j\Big\|_{L_1(\Omega,X)} \le C \Big\|\sum_{j=1}^n \varepsilon_j x_j\Big\|_{L_1(\Omega,X)}. \tag{6.2}$$

Notice, that for a Hilbert space X, norm boundedness of τ and R-boundedness are the same since $\|\Sigma\varepsilon_j T_j x_j\|_{L_1(\Omega)} \approx (\Sigma\|T_j x_j\|^2)^{1/2}$. A little later we will interpret (6.2) in L_q-spaces. For the properties of R-bounded sets, see [15] and [61, Section 2].

Finally we introduce two important classes of sectorial operators: a sectorial operator is called *R-sectorial* (*almost R-sectorial*), if for some $\omega \in (0, \pi)$ the set $\{\lambda R(\lambda, A) : |\arg\lambda| > \omega\}$ (the set $\{\lambda A R(\lambda, A)^2 : |\arg\lambda| > \omega)\}$ is not just bounded, but R-bounded. The infimum over all ω for which this is true is denoted by $\omega_R(A)$ (or $\omega_r(A)$). R-sectorial operators were introduced since the Cauchy problem $y' + Ay = f$ (for given $f \in L_p(\mathbb{R}_+, X)$ and $y(0) = 0$) has maximal L_p-regularity if and only if $\omega_R(A) < \frac{\pi}{2}$ ([81]). Almost R-sectoriality emerged as a basic property in connection with the H^∞-functional calculus in [54] and [57]. The following theorem is from [54] and [57]:

Theorem 6.1. *Let A be a sectorial operator on a Banach space X. For $\psi \in H_0^\infty(\Sigma_\omega)$ with $\psi \neq 0$ and $\omega > \omega(A)$ consider the conditions*

i) *A has a bounded $H^\infty(\Sigma_\sigma)$ calculus.*
ii) *A is almost R-sectorial with $\omega_r(A) \leq \omega$ and $\|\cdot\|_\psi$ is equivalent to the norm on X.*

Then i) $\Rightarrow$ ii) *if $\omega > \sigma$ and* ii) $\Rightarrow$ i) *if $\sigma > \omega$. Furthermore, we have $\omega_{H^\infty}(A) = \omega_r(A)$.*

Note that we can choose any $\psi \in H_0^\infty(\Sigma_\omega)$ in this statement, although the most common choices are $\psi(\lambda) = \lambda^s(1-\lambda)^{-t}$ with $0 < s < t$ or $\psi(\lambda) = \lambda^\alpha e^{-\lambda}$ if $\alpha > 0$ and $\omega(A) < \frac{\pi}{2}$. The proof is based on the fact for every almost R-bounded operator $A, \omega > \omega_r(A)$, and non-zero $\phi, \psi \in H_0^\infty(\Sigma_\omega)$, there is a constant C such that for all $\sigma > \omega$ and $f \in H^\infty(\Sigma_\sigma)$

$$\|f(A)x\|_\psi \leq C\|x\|_\phi,\ x \in X$$

(cf. [54]; for earlier similar statements in Hilbert- and L_p-spaces, see [70], [67]). We remark that counterexamples by N. Kalton in [53] and [57] show that, even for subspaces of L_p-spaces, one does not have $\omega(A) = \omega_{H^\infty}(A)$. This answers a question in [18].

Example 6.2. If X is an L_q-space then for $x_k \in X$

$$\int_\Omega \Big\| \sum_k \varepsilon_k(\omega) x_k \Big\| d\omega \approx \ \Big\| \Big(\sum_k |x_k(\cdot)|^2 \Big)^{\frac{1}{2}} \Big\|_{L_q(\Omega)}.$$

This follows from inequalities of Kahane and Khinchine, (see, e.g., [61, Section 2]). Condition (6.2) for R-boundedness reads now as follows

$$\Big\| \Big(\sum_k |T_k x_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L_q} \leq C \Big\| \Big(\sum_k |x_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L_q}.$$

Such square function estimates are quite common in harmonic analysis (see, e.g., [78, Section 2]). If $A = -\Delta$, then the equivalence $\|x\|_\psi \approx \|x\|$ also takes on a familiar form: choose $\phi \in C^\infty(\mathbb{R})$ with $\phi \geq 0$, $\operatorname{supp}\phi \subset [\frac{1}{4}, 2)$ and $\sum_{k\in\mathbb{Z}} \phi(2^k t) = 1$ for $t > 0$. Put $\psi(\lambda) = \int e^{i\lambda t}\phi(t)dt$ for $|\arg\lambda| < \frac{\pi}{2}$. Then $\psi \in H^\infty(\Sigma_\omega)$ for $\omega < \frac{\pi}{2}$ and we try to express $\psi(2^k A)$ as a convolution operator. As in example 3.1 we have

$$\psi(2^k A)x = \mathcal{F}^{-1} M_{\psi(2^k|t|^2)} \mathcal{F} x = \phi_k * x$$

where ϕ_k is defined by $\phi_k = \check{\phi}_0(2^{k/2}\cdot)$ with $\phi_0(t) = \phi(|t|^2)$ for $t \in \mathbb{R}^n$. Hence

$$\|x\|_\psi = \Big(\int \Big\| \sum_k \varepsilon_k(\omega)\psi(2^k A)x\Big\|_X d\omega\Big)^{1/2} \approx \Big\|\big(\sum_k |\psi(2^k A)x|^2\big)^{\frac{1}{2}}\Big\|_{L_q}$$
$$= \Big\|\big(\sum_k |\phi_k * x|^2\big)^{\frac{1}{2}}\Big\|_{L_q} \approx \|x\|_{L_q}$$

by the Paley-Littlewood decomposition of $L_q(\mathbb{R})$ with respect to the 'partition of unity' $\hat{\phi_k}$. So for $A = -\Delta$ the 'square function' $\|\cdot\|_\psi$ corresponds to the Paley-Littlewood decomposition and we may regard the equivalence $\|x\| \approx \|x\|_\psi$ of Theorem 6.1. as a generalized form of the Paley-Littlewood decomposition for a sectorial operator with a bounded H^∞-calculus.

In the light of Section 3 the next result (from [55]) gives us many examples of R-bounded sets and R-sectorial operators.

Theorem 6.3. *Let X be a subspace of a space $L_q(\Omega)$ with $1 \le q < \infty$ (or more generally, let X have Pisier's property (α)). If X has a bounded H^∞-calculus, then for $\sigma > \omega_{H^\infty}(A)$ we have that*

$$\{f(A) : f \in H^\infty(\Sigma_\sigma),\ \|f\|_\infty \le 1\}$$

is R-bounded.

In particular, boundedness of the H^∞-calculus of A implies that A is R-sectorial with angle $\omega_{H^\infty}(A)$ (For UMD this was shown in [16] and for spaces X with the weaker property Δ in [55]).

Note. We mention two more conditions that characterize the boundedness of the H^∞-functional calculus of a sectorial operator A. They are

- For $\psi(\lambda) = \lambda^s(1-\lambda)$ with $0 < s < 1$
$$\sup_N \sup_{\delta_k=\pm1} \sup_{t>0} \Big\| \sum_k \delta_k \psi(e^{\pm i\omega} 2^k A)\Big\| < \infty$$
(see [55]).
- The functions $t \in \mathbb{R}_+ \to AR(te^{\pm i\omega}, A)x$ are weakly of bounded variation, i.e., there is a constant $C < \infty$ such that for all $x \in X$ and $x^* \in X^*$
$$\int_0^\infty |\langle AR(te^{\pm i\omega}, A)^2 x, x^*\rangle| dt \le C\|x\|\|x^*\|$$
(see [14] and [18]).

7. Characterization by dilations and imaginary powers

Dilation theorems have been among the early tools for proving the boundedness of the H^∞-functional calculus. E.g. accretive operators on Hilbert spaces were first treated with the help of the Sz.-Nagy dilation theorem for contractions. The result for positive contractive semigroups e^{tA} on $L_q(\mathbb{R}^n)$ in Section 3.2 follows from the existence of a dilation of e^{-tA} to a positive contractive group on a (larger) L_q-space (cf. [38]), the boundedness of the H^∞-calculus of a bounded group on an L_q-space ([50]) and from $\omega_{H^\infty}(A) = \omega_R(A) < \pi/2$. The last equality is again shown by combining Fendler's dilation result with Theorem 6.1 (cf. [55]). It is therefore of interest to explore in which Banach spaces other then the Hilbert spaces (see Section 5) dilations characterize the boundedness of the H^∞-calculus.

Theorem 7.1. *Let A be a sectorial operator with $\omega_r(A) < \pi/2$ on a UMD-space X and T_t the semigroup generated by $-A$. Then A has a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \pi/2$ if and only if there is a bounded group U_t on $L_2(\mathbb{R}, X)$, an isomorphic embedding $J : X \to L_2(\mathbb{R}, X)$ and a projection $P : L_2(X) \to J(X)$ such that*

$$\begin{array}{ccc} X & \xrightarrow{\ J\ } & L_2(X) \\ {\scriptstyle T_t}\big\downarrow & & \big\downarrow{\scriptstyle U_t} \\ X & \xleftarrow{J^{-1}\circ P} & L_2(X) \end{array}$$

commutes for $t > 0$.

If X is a space $L_q(\Omega)$ with $1 < q < \infty$, then $L_2(\mathbb{R}, X)$ can be replaced in this statement by another $L_q(\Omega')$ space.

One direction follows essentially from the fact that a bounded group on a UMD-space has a bounded H^∞-calculus (cf. [50]). The construction of the dilation uses a square function technique similar to the one discussed in Section 6 (see [41]). There is also an analogue of Theorem 5.2 for a larger class of Banach spaces, in particular subspaces of $L_q(\Omega)$ for $1 \le q < \infty$.

To formulate our theorem we need a substitute for normal operators on Hilbert spaces. We call a sectorial operator N on a Banach space X *ω-spectral*, if there is a bounded algebra homomorphism from the bounded Borel functions $\mathcal{B}_b(\partial\Sigma_\omega)$ to the bounded operators on X

$$f \in \mathcal{B}_b(\partial\Sigma_\omega) \to f(A) \in B(X)$$

which extends to H^∞-functional calculus and has the convergence property

- If $\|f_n\|_\infty \le C$ and $f_n(\lambda) \to f(\lambda)$ for all $\lambda \in \partial\Sigma_\omega$ then $f_n(A)x \to f(A)x$ for all $x \in X$.

For $\omega = \pi/2$, ω-spectral operators are precisely the operators of scalar type (see, e.g., [30, Section XVIII Thm. 11]) ω-spectral operators have the same spectral theory as normal operators, but they are rare in non-Hilbert spaces because of the general lack of spectral projections in Banach spaces. However, in [41] the following result is shown

Theorem 7.2. *A sectorial operator on a uniformly convex Banach space X has a bounded H^∞-calculus if and only if there is a ω-spectral operator N on another Banach space Y, an isomorphic embedding $J : X \to Y$ and a bounded projection*

$P : Y \to J(X)$ such that

$$\begin{array}{ccc} X & \xrightarrow{J} & L_2(X) \\ {\scriptstyle f(A)}\Big\downarrow & & \Big\downarrow{\scriptstyle f(N)} \\ X & \xleftarrow{J^{-1}\circ P} & Y \end{array} \qquad f \in H^\infty(\Sigma_\sigma)$$

commutes. If $\omega_r(A) < \pi/2$, then we can choose N as a $\pi/2$-spectral operator which generates a bounded group with $e^{-tA} = (J^{-1} \circ P)e^{-tN} J$ for $t > 0$.

We have seen in Section 5 that in a Hilbert space bounded imaginary powers imply the boundedness of the H^∞-calculus. This is not true anymore in general Banach spaces.

Example 7.3. Let $X = L_p(\mathbb{R})$ and A be the unbounded Fourier multiplier operator $Ax = \mathcal{F}^{-1}[m\hat{x}]$ with $m(t) = e^t$ and maximal domain in X. Then $A^{is}x = \mathcal{F}^{-1}[a_s\hat{x}]$ with $a_s(t) = e^{ist}$, i.e., A^{is} is the translation group on $L_p(\mathbb{R})$. However, it is shown in [18] that A does not have a bounded H^∞-calculus for $p \neq 2$.

The next theorem from [57] explains, what additional property the operators A^{it} must have, so that the H^∞-calculus of A will be bounded.

Theorem 7.4. *Let A be a sectorial operator on a uniformly convex Banach space X. If for some $T > 0$ the set $\{A^{it} : -T < t < T\}$ is R-bounded, then A has a bounded H^∞-calculus. If X is a subspace of a space $L_q(\Omega)$ with $1 < q < \infty$, then the converse is true.*

If the operators A^{it} are bounded they form a group with generator $N = i \log A$. N is an operator of *strip type*, i.e., $\sigma(A) \subset S_a = \{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| < a\}$ and $R(\lambda, N)$ is bounded outside of S_a. For such operators one can construct an $H^\infty(S_b)$-calculus along the lines of Section 2 (see [13], [47]) and prove a version of 7.4 for general C_0-groups of operators (see [56], [57]), which generalizes a Hilbert space result of [13].

Theorem 7.5. *Let N be an operator of strip type on a uniformly convex Banach space. If N generates a C_0-semigroup on X such that $\{T_t : -T < t < T\}$ is R-bounded for some $T > 0$, then N has a bounded $H^\infty(S_a)$-calculus for some $a > 0$. If X is a subspace of an $L_p(\Omega, \mu)$-space with $1 < p < \infty$, then the converse is true.*

Of particular interest is the case of bounded groups: here we obtain an extension of Stone's theorem to the Banach space setting (see [56], [57]).

Corollary 7.6. *Let N be an operator of strip type on a uniformly convex Banach space X. If N generates a C_0-group with $\{T_t : t \in \mathbb{R}\}$ R-bounded, then N is $\pi/2$-spectral. The converse is true, e.g., for subspaces of $L_p(\mu)$-space with $1 < p < \infty$.*

It is well known that $N = \frac{d}{dx}$ is not a spectral operator on $L_p(\mathbb{R})$ for $p \neq 2$. This corresponds to the well-known fact that the translation semigroup on $L_p(\mathbb{R})$ is not R-bounded for $p \neq 2$.

Note. The Theorems 7.3, 7.4, 7.5 and Corollary 7.6 are shown in [56], [57] for larger classes of spaces. E.g. instead of uniformly convex Banach spaces we can consider spaces of finite cotype. In the reverse implication, subspaces of $L_p(\Omega)$ can be replaced by spaces with Pisier's property (α). Furthermore, the use of square functions allows to prove more precise, equivalent statements.

8. Perturbation theorem for the H^∞-calculus

There are three basic perturbation results for a sectorial operator A and a bounded operator $B : D(A) \to X$ usually formulated for generators of analytic semigroups, (see, e.g., [74], [37])

i) For a constant C (depending on A) such that for *small perturbations* B with
$$\|Bx\| \leq C\|Ax\|, \; x \in D(A) \tag{8.1}$$
the operator $A + B$ with $D(A + B) = D(A)$ is again sectorial.

ii) If $B : D(A) \to X$ is a *compact perturbation*, then there is a $\lambda > 0$ (depending on A and B) such that $\lambda + A + B$ is a sectorial operator.

iii) If $0 \in \rho(A)$ and B is a *lower order perturbation*, i.e., $B : D(A^\alpha) \to X$ is bounded for some $\alpha \in (0, 1)$, then there is a $\lambda > 0$ such that $\lambda + A + B$ is a sectorial operator.

These results extend to R-sectorial operators and these perturbation results account for some of the usefulness of R-boundedness in establishing maximal regularity. But for the boundedness of the H^∞-calculus only a statement of the type iii) holds (cf. [4], see also [61], Section 13).

Proposition 8.1. *Let A have a bounded H^∞-calculus on X and assume that $0 \in \rho(A)$. Let $\delta \in (0, 1)$ and suppose that B is a linear operator in X satisfying $D(B) \supset D(A)$ and*
$$\|Bx\| \leq C\|A^\delta x\|, \; x \in D(A)$$
then $\lambda + A + B$ has a bounded H^∞-calculus for $\lambda \geq 0$ sufficiently large.

Perturbation theorems of type i) or ii) fail for the H^∞-calculus in general (cf. [72]) and this accounts for much of the difficulties one encounters in proving the boundedness of the H^∞-calculus for differential operators. What assumptions do we have to add to (8.1) or the compactness of B to obtain valid perturbation theorems? Before we answer this question we discuss a result from [54] which shows that the information on the H^∞-calculus is 'encoded' in the fractional domain spaces of a sectorial operator.

Theorem 8.2. *Suppose that A admits a bounded $H^\infty(\Sigma_\sigma)$ calculus on a Banach X and B is an almost sectorial operator on X with $\omega_r(B) < \sigma$. Assume that for two different $u_1, u_2 \in \mathbb{R} \setminus \{0\}$ we have for $j = 1, 2$*
$$D(A^{u_i}) = D(B^{u_i}) \text{ and } \|A^{u_i}x\| \approx \|B^{u_i}x\| \text{ for } x \in D(A^{u_i}) \tag{8.2}$$
then B has a bounded $H^\infty(\Sigma_\nu)$-calculus for $\nu > \sigma$.

Remark 8.3. If u is negative, condition (8.2) amounts to a range condition of the kind $\mathcal{R}(A^v) = \mathcal{R}(B^v)$ with $v = -u > 0$. For reflexive spaces X this can be rephrased by domain conditions on A^* and B^*, i.e., $D((A^*)^v) = D((A^*)^v)$. Indeed, for a reflexive space we have

$$\begin{aligned} &\|B^{-v}x\| \le C\|A^{-v}x\| \text{ for } x \in D(A^{-v}) \\ &\quad \Leftrightarrow \ \|(A^*)^v x^*\| \le C\|(B^\alpha)^v x^*\| \text{ for } x^* \in D((B^*)^v). \end{aligned} \tag{8.3}$$

For a Hilbert space X the case $D(A) \approx D(B)$ and $D(A^*) \approx D(B^*)$ is already contained in [9] and motivated the result above.

Examples 8.4. Theorem 8.2 allows to 'transfer' the boundedness of the H^∞-calculus from a 'simple' operator, such as $A = -\Delta$ on $X = L_p(\mathbb{R}^n)$ (see 3.1) to a more 'complicated' operator B, such as an elliptic differential operator as in (3.3) of Section 3.4. Indeed, if B has constant coefficients, or coefficients smooth enough (e.g., $a_\lambda \in C_b^N(\mathbb{R}^n)$) so that $D(B) = D(A) = \dot{H}_p^{2m}(\mathbb{R}^n)$ and $\mathcal{D}(B') = \mathcal{D}(A') = H_q^{2m}(\mathbb{R}^n)$, $1/p + 1/q$, with equivalent norms then 8.2. applies with $u_1 = 1$ and $u_2 = -1$.

Another consequence of Theorem 8.2. concerns operators A and B, where A has a bounded H^∞-calculus and B has bounded imaginary powers. Since in this case can apply 4.1. to both operators, we need (8.2) only for one $u \neq 0$, e.g., $u = 1$, to conclude that B has a bounded H^∞-calculus.

We want to give an indication how Theorem 8.2 follows from the characterization of the boundedness of the H^∞-calculus in Section 5.

Notation 8.5. For a sectorial operator A we denote by $\dot{X}_\alpha$ the completion $D(A^\alpha)$ with respect to the norm $\|x\|_\alpha = \|A^\alpha x\|$. Note that they agree with the usual 'Sobolev towers' X_α of A as defined, e.g., in [37] only when $0 \in \rho(A)$. Also $\|\cdot\|_1$ only equals the graph norm if $0 \in \rho(A)$. For $A = -\Delta$ on $L_p(\mathbb{R}^n)$, the spaces $\dot{X}_\alpha$ are the Riesz potential spaces $\dot{H}_p^{2\alpha}(\mathbb{R}^n)$, whereas for $A = I - \Delta$ the spaces X_α agree with the Bessel potential spaces $H_p^{2\alpha}(\mathbb{R}^n)$ (cf. [77]). If we want to emphasize the underlying operator A we write $\dot{X}_\alpha = \dot{X}_{\alpha,A}$

In addition define operators $\dot{A}_\alpha$ on $\dot{X}_\alpha$ as follows. Since $A^\alpha : D(A^\alpha) \to R(A^\alpha)$ extends to an isomorphism $\widetilde{A^\alpha} : X_\alpha \to X$ we can define a sectorial operator $\dot{A}_\alpha = (\widetilde{A^\alpha})^{-1} A \widetilde{A^\alpha}$ on $\dot{X}_\alpha$ similar to A, whose resolvent can be obtained as the extension of $R(\lambda, A)$ from the invariant subspace $D(A^\alpha)$ to its completion $\dot{X}_\alpha$.

Proof of Theorem 8.2 (Sketch). Let us assume for simplicity that $u_1 = u, u_2 = -u$ for some $0 < u < 1$ and that $S = B^u A^{-u}$, $T = B^{-u} A^u$ extend to bounded operators on X. We will use the square functions $\rho_{r,s}(\lambda) = \lambda^r (1-\lambda)^{-s}$ with $0 < r < s$ so that

$$\rho_{r,s}(\lambda^{-1}A) = \lambda^{s-r} A^r R(\lambda, A)^s.$$

To compare $\rho_{r,s}(\lambda^{-1}A)$ and $\rho_{r,s}(\lambda^{-1}B)$ we start from the resolvent equation

$$R(\lambda, B) = R(\lambda, A) + R(\lambda, B) B R(\lambda, A) - R(\lambda, B) A R(\lambda, A)$$

choose $v \in (u,1)$ and multiply with $\lambda^{2-v}B^v R(\lambda,B)$. Then

$$\begin{aligned}\lambda^{2-v}B^v R(\lambda,B)^2 = {} & [\lambda^{1-v+u}B^{v-u}R(\lambda,B)]S[\lambda^{1-u}A^u R(\lambda,A)] \\ & + [\lambda^{1-v+u}B^{1+v-u}R(\lambda,B)^2]S[\lambda^{1-u}A^u R(\lambda,A)] \\ & - [\lambda^{2-u-v}B^{u+v}R(\lambda,B)^2]T[\lambda^u A^{1-u}R(\lambda,A)]\end{aligned}$$

or

$$\begin{aligned}\rho_{v,2}(\frac{1}{\lambda}B) = {} & \rho_{v-u,1}(\frac{1}{\lambda}B)S\rho_{u,1}(\frac{1}{\lambda}A) \\ & + \rho_{1+v-u,2}(\frac{1}{\lambda}B)S\rho_{u,1}(\frac{1}{\lambda}A) + \rho_{u+v,2}(\frac{1}{\lambda}B)T\rho_{1-u,1}(\frac{1}{\lambda}A).\end{aligned}$$

Since B is almost R-bounded one can show that the sets $\{\rho_{r,s}(tB) : t \in \mathbb{R}_+\}$ are R-bounded for $0 < r < s$. Hence with the norm

$$\|x\|_{B,\rho_{r,s}} = \Big\| \sum_{k\in\mathbb{Z}} \epsilon_k(\cdot)\rho_{r,s}(2^k B)x \Big\|_{L_2(\Omega,X)}$$

defined as in the beginning of Section 6 we find that

$$\|x\|_{B,\rho_{v,2}} \le C_1\|x\|_{A,\rho_{u,1}} + C_2\|x\|_{A;\rho_{1-u,1}}.$$

Since we can exchange A and B in this argument, the claim of 8.2. follows now from 6.1. and (5.3) of Section 6. $\square$

With a similar technique one can examine the question: when do the fractional domain and range spaces of two sectorial operators A and B coincide, i.e., when do we have for some $\alpha_0, \beta_0 \in (0,1)$ and all $0 < \alpha \le \alpha_0$, $0 < \beta \le \beta_0$

$$\begin{aligned}&\mathcal{D}(A^\alpha) = \mathcal{D}(B^\alpha),\ \|A^\alpha x\| \approx \|B^\alpha x\| \text{ for } x \in \mathcal{D}(A^\alpha) \\ &\mathcal{R}(A^\beta) = \mathcal{R}(B^\beta),\ \|A^{-\beta}x\| \approx \|B^{-\beta}x\| \text{ for } x \in \mathcal{R}(A^\beta) \\ &\mathcal{D}((A^*)^\beta) = \mathcal{D}((B^*)^\beta),\ \|(A^*)^\beta x\| \approx \|(B^*)^\beta x^*\| \text{ for } x^* \in \mathcal{D}((A^*)^\beta).\end{aligned} \tag{8.4}$$

The following theorem is of particular interest if $X_{\alpha,A}$ is a familiar scale of Sobolev spaces, e.g., if $X = L_p(\mathbb{R}^n)$ and $A = -\Delta$. Then (8.4) can be characterized by comparison of A and B on the Sobolev scale.

Theorem 8.6. *Let A have a bounded H^∞-calculus on a reflexive Banach space X and let B be a R-sectorial operator on X. Assume also that $0 \in \rho(A) \cap \rho(B)$ and $\mathcal{D}(B) \subset \mathcal{D}(A^\alpha)$. Then for $\alpha_0, \beta_0 \in (0,1)$ with $\alpha_0 + \beta_0 > 1$, (8.4) holds if and only if for every $\rho \in (1-\alpha_0, \beta_0)$ there is an operator $\widetilde{B}$ on $X_{-\rho,A}$, whose resolvent is consistent with $R(\lambda, B)$ for $\lambda < 0$, such that*

$$\mathcal{D}(\dot{A}_{-\rho}) = \mathcal{D}(\widetilde{B}),\ \|\widetilde{B}x\|_{X_{-\rho,A}} \approx \|\dot{A}_{-\rho}x\|_{X_{-\rho,A}} \quad \textit{for} \quad x \in \mathcal{D}(\dot{A}_{-\rho}).$$

In particular B has a H^∞-calculus with $\omega_{H^\infty}(B) \le \min(\omega_{H^\infty(A)}, \omega_r(B))$.

Theorem 8.4 is a special case of Theorem 5.8 in [54] where $0 \in \sigma(A) \cup \sigma(B)$ is allowed and further equivalent conditions are given. It is motivated by a similar result of Yagi [82] in the Hilbert space case.

Example 8.7. Let $A = -\Delta$ on $X = L_p(\mathbb{R}^n), 1 < p < \infty$, and let B an elliptic differential operator with Hölder continuous coefficients. The method used in [60] to prove R-sectoriality for B on $L_p(\mathbb{R}^n)$ works also on part of the negative Sobolev scale $W_p^s(\mathbb{R}^n)$, and an appeal to 8.3 gives the boundedness of the H^∞-calculus for B. Alternatively one can use perturbation results as in [4].

Now we turn to perturbation results of the type (ii) or (iii) involving 'smallness' of the perturbation or compactness. Our results so far indicate that if we add to (8.1) a second relative boundedness on a fractional domain we may obtain a valid perturbation theorem for the H^∞-calculus. The next statement is from [54, Section 6]; the case $0 < \delta < 1$ was shown independently in [21].

Theorem 8.8. *Let A be an R-sectorial operator on a Banach space X with a bounded $H^\infty(\Sigma_\sigma)$ calculus. Let $0 < |\delta| < 1$ and assume that K is a linear operator with $\mathcal{D}(A) \subset \mathcal{D}(K)$, $\mathcal{R}(K) \subset \mathcal{R}(A^\delta)$*

$$\|Kx\| \le C_0 \|Ax\| \text{ for } x \in \mathcal{D}(A)$$

$$\|Kx\|_{X_{A,\delta}} \le C_1 \|Ax\|_{X_{A,\delta}} \text{ for } x \in \mathcal{D}(A^\delta).$$

If C_0 is sufficiently small and C_1 is finite, then $A+K$ with $\mathcal{D}(A+K) = \mathcal{D}(A)$ has a bounded H^∞-calculus with $\omega_{H^\infty}(A+K) \le \sigma$.

Even *one* relative bound on a fractional domain is enough if one considers the perturbation K not as a map $K : D(A) \to X$ but rather as map in the fractional scale $K : D(A^\alpha) \to D(A^{\alpha-1})$ for some $\alpha \in (0,1)$. However then it becomes more difficult to determine the domain of the perturbed operator.

Theorem 8.9. *Let A be an R-sectorial operator on a Banach space X with a bounded H^∞-calculus. Let $\delta \in (0,1)$ and assume that $K : X_\delta \to X_{\delta-1}$ is a bounded linear operator. If $\|K\|$ is small enough then there exists a unique sectorial operator C in X whose resolvents are consistent with those of $\dot{A}_{\delta-1} + K$ on $\dot{X}_{\delta-1}$ and $\omega_{H^\infty}(C) \le \omega_{H^\infty}(A)$.*

If X is reflexive then the condition $\|K\|_{\dot{X}_\delta \to \dot{X}_{\delta-1}} \le C$ can also be expressed as follows

$$|\langle Kx, x^*\rangle| \le C \|A^\delta x\| \cdot \|(A^*)^{1-\delta} x^*\| \quad \text{ for } x \in \mathcal{D}(A^\delta),\ x^* \in \mathcal{D}((A^*)^{1-\delta}). \quad (8.5)$$

This condition is in the spirit of form perturbation theorems on Hilbert spaces, and indeed it can be applied to Schrödinger operators.

Example 8.10. Let $X = L_p(\mathbb{R}^n)$ and $V : \mathbb{R}^n \to \mathbb{C}$ a potential of the Kato class. By definition this means that $\|\,|V|(\lambda - \Delta)^{-1}\|_{L_1 \to L_1} \to 0$ for $\lambda \to \infty$. By duality and interpolation one can show that

$$\|(\lambda - \Delta)^{-1/p'} V (\lambda - \Delta)^{-1/p}\|_{L_p \to L_p} \to 0 \qquad \text{for } \lambda \to \infty. \quad (8.6)$$

Hence, if $A_\lambda = \lambda - \Delta$ and K is the multiplication operator with V, then (8.6) implies that the norm of $K : X_{\frac{1}{p}, A_\lambda} \to X_{\frac{1}{p}-1, A_\lambda}$ is small if λ is large, where $X_{\alpha, A_\lambda} \approx H_p^{2\alpha}(\mathbb{R}^n)$. It follows from 8.6 that the Schrödinger operator $-\Delta + V + \mu$

has a bounded H^∞-calculus on $H_p^\beta(\mathbb{R}^n)$ for all $\beta \in (-\frac{2}{p^*}, \frac{2}{p})$ and μ sufficiently large.

Finally we address the question of how we have to strengthen the compactness assumption in perturbation statement ii) to obtain a perturbation result for the H^∞-calculus.

Recall that the *approximation numbers* $a_n(T)$ of a bounded operator $T : Y \to X$ are defined by

$$a_n(T) = \inf\{\|T - S\|_{Y\to X} : S \in B(Y,X),\ \dim R(S) < n\}.$$

If X has the approximation property, then T is compact if and only if $a_n(T) \to 0$ for $n \to \infty$. We will have to require that $a_n(T)$ goes to zero fast enough.

Theorem 8.11 ([57]). *Let A have a bounded H^∞-calculus on a Banach space X, that does not contain a complemented copy of c_0 (i.e., X is reflexive). Let $K : D(A) \to X$ be compact and let $a_n = a_n(T : X_1 \to X)$. If in addition*

$$\sum_{n=1}^{\infty} \frac{a_n}{n}(1 + \ln n) < \infty, \tag{8.7}$$

then $A + B$ has a bounded H^∞-calculus.

If X is a Hilbert space, (8.7) can be weakened to $\sum_{n=1}^\infty \frac{a_n}{n} < \infty$. In particular, if K belongs to one of the Schatten classes $S_p(X_1, X)$, then the perturbation theorem applies.

9. H^∞-functional calculus and interpolation

We have seen in 5.3 (cf. [9]) that a sectorial operator on a Hilbert space has a bounded H^∞-calculus if and only if we have for the complex interpolation method

$$\dot{X}_\gamma \approx [\dot{X}_\alpha, \dot{X}_\beta]_\theta, \qquad \text{where } \gamma = (1-\theta)\alpha + \theta\beta \tag{9.1}$$

for some $\alpha < \beta$ and $\theta \in (0,1)$. This does not generalize to Banach spaces, since, as mentioned in 4.1, we only need bounded imaginary powers to obtain (9.1). It is also known, that (9.1) does not hold for the real interpolation method, for all sectorial operators on $L_p(\Omega)$, $1 < p < \infty$, whose resolvents are dominated by integral operators (cf. [59]). Therefore we have to consider a new interpolation method which is well suited for the Rademacher averages appearing in the square functions of Section 6 and in Theorem 6.1. This method will allow us to give further criteria for the boundedness of the H^∞-calculus.

Definition 9.1. *Let (Y_0, Y_1) be an interpolation couple. For every $\theta \in (0,1)$ the Rademacher interpolation space $\langle Y_0, Y_1\rangle_\theta$ consists of all $y \in Y_0 + Y_1$ which can be represented as a sum*

$$y = \sum_{k=-\infty}^{\infty} y_k, \qquad y_k \in Y_0 \cap Y_1 \tag{9.2}$$

convergent in $Y_0 + Y_1$ *such that for* $j \in \{0, 1\}$

$$C_j(y_k) = \sup_N \Big\| \sum_{k=-N}^{N} \epsilon_k 2^{(j-\theta)k} y_k \Big\|_{L_1(Y_j)} < \infty.$$

The norm of $\langle Y_0, Y_1 \rangle_\theta$ *is given by*

$$\|y\|_\theta = \inf \max_{j=0,1} C_j(y_k)$$

where the infimum is taken over all representations (9.2) *of* y.

The Rademacher interpolation method is introduced and studied in [54] and [57]. Here we just mention two facts:

- $\langle L_{p_0}, L_{p_1} \rangle_\theta = L_p$ for $1 \le p_0 < p_1 < \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.
- For an interpolation couple (X, Y), where X and Y are uniformly convex Banach spaces, we have $\langle X, Y \rangle_\theta^* = \langle X^*, Y^* \rangle_\theta$.

The next two theorems are special cases of results in [54, Section 7] where more general Banach spaces are considered.

Theorem 9.2. *Let* A *be an almost* R*-sectorial operator on a uniformly convex Banach space. Then* A *has a bounded* H^∞*-calculus if and only if for some (all)* $\alpha < \beta$ *and* $\theta \in (0, 1)$ *we have* $\dot{X}_\gamma \approx \langle \dot{X}_\alpha, \dot{X}_\beta \rangle_\theta$*, where* $\gamma = (1 - \theta)\alpha + \theta\beta$.

There is an analogue to Dore's result ([23], [24]) for the real interpolation method, which was mentioned in 3.7.

Theorem 9.3. *Let* A *be an almost* R*-sectorial operator on a uniformly convex Banach space. For* $\alpha < \beta$ *and* $\theta \in (0, 1)$ *we have that* A *has an* H^∞*-calculus on* $\langle \dot{X}_\alpha, \dot{X}_\beta \rangle_\theta$ *for* $\sigma > \omega_r(A)$.

Example 9.4. Let A be an elliptic operator of order $2m$ on $L_p(\mathbb{R}^n)$ which is almost R-sectorial and satisfies $X_1 \approx W_p^{2m}(\mathbb{R}^n)$. Since $\mathcal{D}(\Delta^m) \approx W_p^{2m}(\mathbb{R}^n)$ and Δ has a bounded H^∞-calculus we have

$$\langle L_p, W_p^{2m} \rangle_\theta \approx [L_p, W_p^{2m}]_\theta \approx W_p^{2m\theta}.$$

Hence, by 9.3 we obtain operators $\tilde{A}$ on $W_p^s(\mathbb{R}^n)$ for $s \in (0, 2m)$ which are consistent with A and have a bounded H^∞-calculus without assuming (or proving) that $\mathcal{D}(A^\theta) = W_p^{2m\theta}$.

In Section 8 we showed that one can transfer the boundedness of the H^∞-calculus of an operator A to an almost R-sectorial operator B if $\mathcal{D}(A^{u_j}) = \mathcal{D}(B^{u_j})$ with equivalent norms for two different $u_1, u_2 \in \mathbb{R} \setminus \{0\}$. With the help of the Rademacher interpolation method one can weaken this criterion for operators A that are defined on an interpolation scale, e.g., consisting of L_p-spaces, Hardy spaces, Sobolev spaces or Helmholtz spaces. Then it is possible to check the two conditions for u_1 and u_2 in different spaces of the scale, and it may simplify things if one of these spaces can be chosen to be a Hilbert space. Furthermore, one can allow that B is not defined on the same space as A but rather on a complemented subspace of X.

Theorem 9.5 ([54]). *Let (X_0, X_1) be an interpolation couple of uniformly convex Banach spaces with complemented subspaces (Y_0, Y_1) and associated projections P_0, P_1 compatible with the interpolation couple. Assume that A has a bounded H^∞-calculus on X_0 and X_1 and B is almost R-sectorial on Y_0 and Y_1. If there are $\alpha < 0 < \beta$ such that*

$$P_0((X_0)^{\cdot}_{\alpha,A}) = (Y_0)^{\cdot}_{\alpha,B}, \qquad P_1((X_1)^{\cdot}_{\beta,A}) = (Y_1)^{\cdot}_{\beta,B} \tag{9.3}$$

then B has a bounded H^∞-calculus on all complex interpolation spaces $Y_\theta = [Y_0, Y_1]_\theta$.

To be precise we recall that a bounded operator A is *consistent* for (X_0, X_1) if there are $A_1 \in B(X_1)$ and $A_2 \in B(X_0)$ so that $A_1 x = A_2 x$ for $x \in X_0 \cap X_1$. If A is sectorial, we require sectorial operators A_j on X_j such that $R(\lambda, A_j)$ are compatible for $\lambda < 0$. In (9.3) we mean that $P_j : X_j \to Y_j$ restricted to $X_j \cap \dot{X}_{\gamma,A} = \mathcal{D}(A^\gamma)$ has a continuous extension $\tilde{P} : \dot{X}_{\gamma,A} \to \dot{Y}_{\gamma,B}$ which is surjective.

We illustrate 9.4 by an application to Stokes operators ([54, Section 10.e]).

Example 9.6. Let Ω be a bounded domain in $\mathbb{R}^n$ with a $C^{1,1}$-boundary $\partial\Omega$. For $p \in (1, \infty)$ we put $X_p = L_p(\Omega)^n$ and denote by Y_p the Helmholtz space $L_{p,\sigma}(\Omega)^n$. A_p denotes the Dirichlet Laplacian $-\Delta_D$ on X_p and B_p denotes the Stokes operator $P_p A_p$ on Y_p, where P_p is the Helmholtz projection of X_p onto Y_p. One can show that P_p extends to a projection from X_{-1,A_p} onto Y_{-1,B_p} for all $1 < p < \infty$ ($\alpha = -1$), but also to a projection from X_{s,A_p} onto Y_{s,B_p} for $s \in (0, 1/4)$ if $p = 2$. Since it is also known ([40]) that B_p is R-sectorial on Y_p for $1 < p < \infty$, Theorem 9.5 implies that B_p has a bounded H^∞-calculus on Y_p for $1 < p < \infty$.

10. Joint functional calculus and operator sums

In this section we discuss the boundedness of the joint functional calculus of commuting sectorial operators and its applications to sums of commuting sectorial operators. However, to have a convenient tool in hand, we first extend the functional calculus to holomorphic operator-valued functions. The idea of such a calculus goes back to [1] in the Hilbert space setting. The following general formulation is from [55].

Given a sectorial operator A on X, we denote by $\mathcal{A}$ the subalgebra of $B(X)$ of all bounded operators, that commute with resolvents of A. Then we denote by $RH^\infty(\Sigma_\sigma, \mathcal{A})$ the space of all bounded analytic functions $F : \Sigma_\sigma \to \mathcal{A}$ with the additional property that the range $\{F(\lambda) : \lambda \in \Sigma_\sigma\}$ is R-bounded in $B(X)$. The norm for an element $F \in RH^\infty(\Sigma_\sigma, \mathcal{A})$ is given by

$$\|F\| = R(\{F(\lambda) : \lambda \in \Sigma_\sigma\}).$$

By $RH_0^\infty(\Sigma_\sigma, \mathcal{A})$ we denote elements of $RH^\infty(\Sigma_\sigma, \mathcal{A})$ which are dominated in operator norm by a multiple of $|\lambda|^\epsilon\,(1+|\lambda|)^{-2\epsilon}$ for some $\epsilon > 0$. For $F \in RH_0^\infty(\Sigma_\sigma, \mathcal{A})$ the integral

$$F(A) = \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} F(\lambda)R(\lambda, A)\,d\lambda \tag{10.1}$$

exists as a Bochner integral for $\omega(A) < \nu < \sigma$. We show now that (10.1) can be extended to a bounded functional calculus on $RH^\infty(\Sigma_\sigma, \mathcal{A})$, if A has a bounded H^∞-calculus.

Theorem 10.1. *Assume that A has a bounded $H^\infty(\Sigma_{\sigma'})$-calculus. Then for all $\sigma > \sigma'$ there is a bounded algebra homomorphism $F \in RH_0^\infty(\Sigma_\sigma, \mathcal{A}) \to F(A) \in B(X)$ with* (10.1) *for $F \in RH_0^\infty(\Sigma_\sigma, \mathcal{A})$ and the following convergence property:*

> *If (F_n) is a bounded sequence in $RH^\infty(\Sigma_\sigma, \mathcal{A})$ and $F_n(\lambda)x \to F(\lambda)x$ for some $F \in RH^\infty(\Sigma_\sigma, \mathcal{A})$ and all $\lambda \in \Sigma_\sigma$ and $x \in X$, then $F_n(A)x \to F(A)x$ for all $x \in X$ as $n \to \infty$.*

Furthermore, if X is a subspace of an $L_p(\Omega)$-space with $1 \le p < \infty$ or, more generally, if X has Pisier's property (α), then for an R-bounded set $\tau \subset B(X)$ the set

$$\{F(A) : F \in RH^\infty(\Sigma_\sigma, \mathcal{A}) \text{ with } F(\lambda) \in \tau \text{ for } \lambda \in \Sigma_\sigma\} \tag{10.2}$$

is R-bounded.

One consequence of Theorem 10.1 is Theorem 4.2 on operator sums. Indeed, by Theorem 10.1 the operator $A(A+B)^{-1} = F(B)$, $F(\lambda) = \lambda(\lambda - A)^{-1}$ is bounded since $F \in RH^\infty(\Sigma_\sigma, \mathcal{B})$ by the assumptions of Theorem 4.2. As a further consequence we obtain the boundedness of the joint functional calculus. For Hilbert spaces this was worked out in [1], for more general Banach space, e.g., spaces with property (α), in [62] and [55]. The following formulation is from [61].

Denote by $H^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n})$ the space of all bounded analytic functions of n variables on $\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n}$ and by $H_0^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n})$ the subspace of functions $f \in H^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n})$ which obey an estimate of the form

$$\|f(\lambda_1, \dots, \lambda_n)\| \le C \prod_{k=1}^n |\lambda_k|^\epsilon\,(1+|\lambda_k|)^{-2\epsilon}$$

for some $\epsilon > 0$. For $f \in H_0^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n})$ we define the joint functional calculus by

$$f(A_1, \dots, A_n) = \Big(\frac{1}{2\pi i}\Big)^n \int_{\partial\Sigma_{\nu_1}} \cdots \int_{\partial\Sigma_{\nu_n}} f(\lambda_1, \dots, \lambda_n) \prod_{j=1}^n R(\lambda_j, A_j)\,d\lambda_1 \dots d\lambda_n \tag{10.3}$$

where $\omega(A_i) < \nu_i < \sigma_i$ for $i = 1, \dots, n$. If $n = 2$ and Φ_1 denotes the operator valued functional calculus of A_1 then we can rewrite (10.3) by Fubini's theorem as

$$\begin{aligned} f(A_1, A_2) &= \frac{1}{2\pi i} \int_{\partial\Sigma_{\nu_1}} \Big(\frac{1}{2\pi i} \int_{\partial\Sigma_{\nu_2}} f(\lambda_1, \lambda_2) R(\lambda_2, A_2)\, d\lambda_2 \Big) R(\lambda_1, A_1)\, d\lambda_1 \\ &= \int_{\partial\Sigma_{\nu_1}} f(\lambda_1, A_2) R(\lambda_1, A_1)\, d\lambda_1 \\ &= \Phi_1(f(\cdot, A_2)). \end{aligned}$$

This indicates already how to construct the joint functional calculus of n operators by induction.

Theorem 10.2. *Suppose that X is a subspace of an L_p-space with $1 \le p < \infty$ (or more generally a space with property (α)). Assume that $A_1, \dots, A_n$ are resolvent commuting operators with a bounded $H^\infty(\Sigma_{\sigma'_j})$ functional calculus for $j = 1, \dots, n$, respectively. Then for $\sigma_j > \sigma'_j$ we have an algebra homomorphism $f \in H^\infty(\Sigma_{\sigma_1} \times \dots \times \Sigma_{\sigma_n}) \to f(A_1, \dots, A_n) \in B(X)$ extending (10.3) with the following convergence property:*

If $\{f_m, f\}$ is uniformly bounded in $H^\infty(\Sigma_{\sigma_1} \times \dots \times \Sigma_{\sigma_n})$ and $f_m(\lambda_1, \dots, \lambda_n) \to f(\lambda_1, \dots, \lambda_n)$ as $m \to \infty$ for all $\lambda_i \in \Sigma_{\sigma_i}$, then $f_m(A_1, \dots, A_n)x \to f(A_1, \dots, A_n)x$ as $m \to \infty$ for all $x \in X$.

Using $f(\lambda_1, \lambda_2) = f(\lambda_1 + \lambda_2)$ we obtain

Corollary 10.3. *Let X, A_1 and A_2 be as in 10.2. If A_1 and A_2 have a bounded H^∞-calculus with $\omega_{H^\infty}(A_1) + \omega_{H^\infty}(A_2) < \pi$ then $A_1 + A_2$ has a bounded H^∞-calculus with $\omega_{H^\infty}(A_1 + A_2) \le \min(\omega_{H^\infty}(A_1), \omega_{H^\infty}(A_2))$.*

Corollary 10.3 also holds for a general Banach space if A_1 has an R-bounded H^∞-calculus (cf. [54]).

Furthermore, there is a non-commutative version of Corollary 8.3 in [75] where the commutativity assumption is replaced by $0 \in \rho(A)$ and the following commutator condition for $0 \le \alpha < \beta < 1$:

$$\left\| \lambda(\lambda + A)^{-1} [A^{-1}(\mu + B)^{-1} - (\mu + B)^{-1} A^{-1}] \right\| \le \frac{C}{|1 + \lambda|^{1-\alpha} \, |\mu|^{1+\beta}} \tag{10.4}$$

for $\lambda \in \Sigma_{\pi - \nu_A}, \mu \in \Sigma_{\pi - \nu_B}$ with $\nu_A > \omega_{H^\infty}(A), \nu_B > \omega_{H^\infty}(B)$ and $\nu_A + \nu_B < \pi$.

Theorem 10.4. *Let A have a bounded H^∞-calculus and B an R-bounded H^∞-calculus on a Banach space X such that $\omega_{H^\infty}(A) + \omega_{RH^\infty}(B) < \pi$. Then there is a constant c_0 so that $A + B$ has a bounded H^∞-calculus if (10.4) holds with $C \le c_0$.*

Note. The (R-)boundedness of the operator-valued H^∞-calculus and the joint functional calculus for bisectorial operators are treated in [55]. In [68] it is shown that 10.3 also holds under the weaker assumption that X has property (Δ). Analogues of 10.1 and 10.2 for operators of strip type on Banach spaces are contained in [56], [57].

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